Chapter Seven

How Can We Know Mathematical Truths?

Abstract: The goal of this chapter is to present a summary sketch of an 'antirealist' account of mathematics as contrasted to opposing 'realist' positions. This chapter suggests that: 1) the inference rules that make up any formal system are prescriptive, 2) the implicit and explicit axioms underlying formal systems are prescriptive, 3) the definitions found in the deductive sciences are stipulative and are thus prescriptive, 4) the 'truths' deduced in formal deductive systems can be understood as 'true-in-a-language,' and 5) a 'game formalist' account complements the predominately externalist (PE) definition of knowledge. A formalist explains how one can know that '141678 + 639465 = 781143'. The method of conceptual analysis will be used to examine the concepts and structure of artificial deductive systems.

How does one come to know a mathematical truth? Most people are comfortable in believing that they know many mathematical truths but cannot explain how these truths are known. In this chapter, I provide an elementary explanation of where 'mathematical truth' fits in among our ordinary thoughts. It is evident that there is a difference in the way that we come to know empirical truths (e.g., 'there is a chair sitting in the corner') as compared to mathematical truths (e.g., '141678 + 639465 = 781143'). In this chapter, we examine mathematics (i.e., arithmetic and geometry) and standard (predicate and propositional) deductive logic. I refer to these as 'formal deductive systems.'[1]

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[1] A terminology review for logic novices: In a 'valid' deductive argument, the structure of the argument makes it so that, *if* the premises are true, then the conclusion must be true. The following is an example of a valid argument:

> Premise: If A, then B.
> Premise: A.
> Conclusion: B.

This argument is valid because if the two premises are true, then the conclusion cannot be false. In other words, if both premises are true, then the conclusion is *necessarily* true, and the argument is 'sound.' But with the insertion of substantive premises, it is (contingently) possible for (at least) one premise to be false (as follows):

> Premise: 1) If **x** is a pig, then it has wings and flies.
> Premise: 2) **x** is a pig.
> Conclusion: 3) **x** has wings and flies.

This argument is valid, since it has the same syntactic and inferential form as the first argument, but it is called an 'unsound argument' because the first premise is false, and it generates a false conclusion. The first sentence asserts that 'all pigs fly' (and that is false). This false first premise and the assumed truth of the second premise entail the conclusion that 'a pig has wings and flies' (and this is false). The concepts of ‘validity’ and ‘logical consequence’ are closely related: an argument is valid just in case its conclusion is a logical consequence of its premises.

In chapter one, we discussed how knowledge is possible. We concluded that knowledge ('**S** knows **p**') is obtained when all four of the predominately externalist (PE) conditions of knowledge are satisfied. With this definition in mind, we will reply to the following three questions with regard to mathematics (and deductive systems in general):

1) What is the *epistemic status* of axioms, definitions, and inference rules in mathematics? Can we *know* them to be *true*?

2) Do mathematical entities (e.g., squares, numbers, and ratios) *exist*? And if so, in what sense (and how) do they exist?

3) What is the source of mathematical truth?

Explaining the nature of deductive systems might seem like (and actually is) a daunting task. Most philosophers will scoff at an attempt of an explanation of logic and mathematics in a single chapter. But I will put aside this pessimism (and the complications of detail) and attempt a *conceptual analysis* of the structure of deductive systems. The analysis is based on insights gained from the prescriptive-descriptive distinction. This chapter seeks to suggest new thoughts about the nature of mathematics and logic, hoping to inspire more rigorous and detailed analyses from philosophers with more expertise in this field. A cursory sketch of 'anti-realist mathematical formalism' will be endorsed here in contrast to 'mathematical realism.'

The Structure of Formal Deductive Systems

In order to answer the three primary questions, we need to understand the structure of formal deductive systems. The purpose of a formal system is to establish a precise language and inference rules that enable one to validly deduce one proposition as entailed from the assumed truth of other proposition(s). These rules are 'truth preserving' if they allow the valid syntactical entailment of assumed-true premises to necessarily true conclusions. An artificial formal system may be formulated and studied for its intrinsic properties (as in pure mathematics) or it may be formulated in terms of a heuristic description (i.e., a model) of external phenomena.[2] Deductive logic, geometry, arithmetic, set theory, probability theory, statistics, formal semantics, and computer languages are all examples of 'formal systems.' The general structure of deductive systems consists of the following elements: 1) the introduction of a *vocabulary* of symbols and

[2] Historically a number of philosophers, including Frege and Russell believed that there is a single logic that underlies all of our reasoning, no matter whether it is about science, mathematics, or philosophy. Most current philosophers, respecting Carnap's work and others, now accept that there are multiple consistent systems of logics.

definitions about what counts as an individual constant, individual variable, predicate, proper name, sentential connective, punctuation, and quantifier, 2) the introduction of *syntactical formation rules* (or grammar) that defines how 'well-formed formulas' are to be constructed out of symbols (i.e. a procedure that determines whether a sentence, as a finite strings of words or symbols, is 'meaningful' or not) 3) a set of truth-preserving *inference rules* (e.g. *modus ponens*), and 4) a *semantics* (i.e. interpretations using symbolization keys and extensions, e.g. 'm' means Mary, the predicate 'A' means asleep). Behind this familiar structure there are implicit *axioms* and *definitions* (discussed below) that underlie formal systems.[3] On the view here, formal deductive systems, including their axioms, vocabulary, formation rules, inference rules, and semantics, are constructed by logicians. Formal deductive systems are 'prescriptive' in that they stipulate rules concerning the regimented use of linguistic expressions. To understand what a formal system is, let's examine the nature of 'vocabulary,' 'syntax,' 'inference rules' and 'axioms.'

Vocabulary and Syntax

The easiest way to explain vocabulary and syntax, is to illustrate with an example:

Vocabulary (Categories and Basic expressions):

Category	*Basic Expressions*
Names	d, n, j, m, n
One-place predicates	M, B, P
Two-place predicates	H, K, L

Syntax

Truth-functional 'connectives' ('not,' 'and,' 'or') are designated and stipulated with a symbol (-, &, v) and a syntax (-p, p & q, p v q).

1) If P is a one-place predicate, and n is a name, then Pn is a sentence.
2) If L is a two-place predicate, and m and n are names, then L(m,n) is a sentence.
3) If S is a sentence, then '-S' is a sentence.
4) If S and T are sentences, then 'S & T' is a sentence.
5) If S and T are sentences, then 'S v T' is a sentence.

[3] From Henry Kyburg, Jr. & Choh Man Teng (2001): There is a canonical procedure for characterizing the language of a formal system. The language consists of a *vocabulary, formation rules, axioms, and rules of inference*. The *vocabulary* consists of predicates (**x** is red, **x** is even), relation expressions (**x** is the brother of **y**, **m** lies between **l** and **n**), operations (the sum of **x** and **y**, the mother of **x**), proper names (John, 25) and an infinite set of variables (**x1**, **x2**, **x3**). …The *formation rules* tell us what a well-formed formula or well-formed term of the calculus is. These rules must be (in ordinary cases) recursive, because we want to allow for an unlimited number of sentences (p. 25).

LTF Gamut (1991, vol 1) state:

> A formal language is characterized by *vocabulary* and *syntax*. The vocabulary of a formal language is what determines the *basic expressions* it contains… In the syntax of a language, a definition is given of the composite expressions of the language. The definition is given in a number of explicit rules which say how expressions may be combined with each other, thus creating other expressions. The principle of compositionality presides over the whole process: the meaning of a composite expression must be wholly determined by the meanings of its composite parts and of the syntactic rules by means of which it is formed (p. 26).

With vocabulary and syntax, a recursive mode of sentence structure is stipulated. Examples of sentences and their symbolic translation are: 'Norman is poor' is represented as 'Pn' and 'Mary loves Norman' is represented as 'L(m,n).' These are well-formed formulas in logical notation.

Inference Rules

What is an inference rule? An inference rule expresses a relation between premise(s) and a conclusion, whereby the conclusion is entailed (or formally derivable, deducible) from the premise(s). In logic, we adopt a number of elementary 'rules of inference' that guarantee the necessity of a conclusion. These rules are based upon the *syntactic structure* of the argument. Syntax is concerned with constructing, or transforming the symbols of a language, independent of the meaning or semantic designation of the symbols. From propositional logic, let us examine an inference rule (where the symbols 'or' and 'not' have already been stipulatively defined by a truth-table). This *inference rule* is called **'disjunctive syllogism'**:

Premise: 1) **p** or **q**.
Premise: 2) not **q**.
Conclusion: 3) **p**.

This argument form is equivalent to the conditional form: If (**p** or **q**) and not **q**, then **p**.

The disjunctive syllogism is an accepted rule of inference in propositional logic. If both premises are true, then the conclusion is necessarily true.[4] Other rudimentary inference rules include *modus ponens* (footnote #1), *modus tollens*, hypothetical syllogism, conjunction

[4] To illustrate the 'disjunctive syllogism' rule of inference, let us add semantics to the symbols, and suppose that **p** stands for the proposition 'Alex is in New York' and **q** stands for the proposition 'Henry is in Paris.' Suppose that the first premise is true: Alex is in New York *or* Henry is in Paris. Suppose that the second premise is true: it is false that Henry is in Paris. Given the assumed truth of both premises, it can be deduced that Alex is in New York.

introduction, simplification (i.e., conjunct elimination), addition (of a disjunct), and constructive dilemma. Using these elementary rules, long and complex deductive arguments can be made. Similarly, arithmetic is learned and practiced with simple syntactic rules of inference using the functional connectives of plus (+), minus (-), multiplication (x), and so on, which are the basis of larger calculations. A set of inference rules should be effective for determining whether any proposition is or isn't, a conclusion derivable in a formal system.

Inference Rules, Vocabulary, and Syntactic Formation Rules are Prescriptions

That the rules of inference of deductive logic provide a normative standard for reasoning is accepted by many philosophers. The purpose of logic is to formulate principles of correct reasoning and to distinguish good reasoning from bad reasoning. In deductive logic, the logician wants to adopt truth-conducive syntactic inference rules (i.e., entailment, transformation rules) since a true conclusion is sought as an output from true premises. An inference rule is prescribed as a means for attaining a (valued) valid argument. Whether an inference rule is truth-conducive isn't always obvious, and if it isn't truth-conducive, a demonstrated example(s) of its failure (or of inconsistencies) will lead to its rejection. In chapter two, we demonstrated how 'epistemic closure' (currently supported by some epistemologists) is not a truth-conducive inference rule by presenting counterexamples. I suggest that *the introduction of a set of symbols* (a vocabulary) and *the syntactical introduction of formation rules* (or grammar) are also prescriptive.

Axioms

Let us now examine another element that lies behind formal deductive systems, axioms. What is an axiom? The most direct way to answer this question is to inspect sets of axioms:

Euclid's Postulates of Geometry[5]

1. A straight line segment can be drawn joining any two points.
2. Any straight line-segment can be extended indefinitely in a straight line.
3. Given any straight line-segment, a circle can be drawn having the segment as a radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect in such a way that the sum of the inner angles on one side is less than two angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is called the 'parallel postulate.'

[5] These five postulates are the core of Euclidean geometry. If the fifth postulate is discarded, the remaining four postulates can generate alternative non-Euclidean geometries.

Peano's Axioms of Arithmetic[6]

1. Zero is a number.
2. If *n* is a number, the successor of *n* is a number.
3. Zero is not the successor of a number.
4. Two numbers of which the successors are equal are themselves equal.
5. If a set S of numbers contains zero and also the successor of every number in S, then every number is in S.

The Axioms of Number[7]

For any numbers, a, b, and c:

1. Commutative Axiom of Addition: a + b = b + a.
2. Associative Axiom of Addition: (a + b) + c = a + (b + c).
3. Commutative Axiom of Multiplication: a x b = b x a.
4. Associative Axiom of Multiplication: (ab)c = a(bc).

5a. 0 + a = a for every number a,
5b. 0 x a = 0 for every number a,
5c. if ab = 0 then either a = 0 or b = 0 or both are 0.

6. There is a unique number 1 such that 1 x a = a for every number a.
9. Distributive Axiom: ab + ac = a(b + c).

Axioms of Probability[8]

1. Chances are always at least zero.
2. The chances that *something* happens is 100%.
3. If two *events* cannot occur at the same time (if they are disjoint or mutually exclusive), the chance that either one occurs is the sum of the chances that each occurs.[9]

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[6] These axioms were made famous by Peano in 1867, but their original construction was likely done by Dedekind.

[7] These axioms are from Morris Kline (1967), pp. 76-79. Kline, a mathematician, describes eleven basic axioms (e.g., the commutative axiom; a + b = b + a) and says "The set of axioms we have just given is not complete; that is, it does not form the logical basis for *all* of the properties of the positive and negative whole numbers, fractions, and irrational numbers. However, the set does provide the logical basis for what is usually done with numbers in ordinary algebra. Moreover, it does give some idea of what the axiomatic basis for mathematical work with numbers amounts to" (p. 79).

[8] With Andrej Kolmogorov's (1933) mathematical specification of these three axioms, the concept of 'event' is taken as primitive. For the sake of axiomatization, 'events' are treated as sets and dealt with in a set-theoretic fashion. Kolmogorov maintained that there is no need to give a univocal definition of events, as the basic notions of geometry, such as 'point' and 'line,' are not defined in axiomatic geometry. Kolmogorov's axioms were met with wide consensus and became the 'standard' calculus of probabilities. However, Kolmogorov's axiomatization, although best known, isn't the only one, nor was it the first to be worked out. Alternative axiomatizations have been suggested, especially for 'conditional probability.' For example, Karl Popper (1934) and Rudolf Carnap (1950) have proposed different axiomatizations of probability. See Maria Galavotti (2005, pp. 52-54) for more details.

[9] For example, consider an experiment that consist of tossing a coin once. The first axiom says that an outcome must be at least zero. The second axiom says that the chance that the coin either lands heads or tails or lands on its edge or doesn’t land at all is 100%. The third axiom says that the chance that the coin either lands heads or lands tails is the sum of the chance that the coin lands heads and that the coin lands tails is the sum of the chance that the coin land heads and the chance that the coin lands tails, because both cannot occur in the same coin toss. All other mathematical truths about probability can be derived from these three axioms.

Miscellaneous Axioms of Metaphysics, Logic, Semantics, and Set Theory

1. The Law of Identity: For any object x, it is necessarily the case that x is identical with x.
2. Leibniz's Law (of the Indiscernibility of Identicals): For any objects x and y, if x is identical with y, then every property of x is a property of y and vice-versa.
3. All objects are either 'concrete' or 'abstract.'
4. Law of Bivalence: Every declarative sentence/statement expressing a proposition (in a domain) has exactly one truth value, either true or false.
5. Law of Excluded Middle: For every statement **p**, either **p** is true, or **p** is false.
6. Law of Non-Contradiction: It cannot be the case that **p** and not-**p**.
7. The Transitivity of Identity: Whenever a=b, and b=c, then a=c.
8. The Null Set Axiom: The empty set exists.
9. Axiom of Extensibility: Two sets are equal if and only if they contain the same elements.

Axioms are the foundational principles that lie behind the exposition of the syntax and semantics of a formal system. In a deductive axiomatic theory, the set of axioms are the basis of a system, while the remaining definitions and propositions (e.g., theorems) are the logical consequence of the axioms. In mathematics, the 'axiomatic method' originated with the works of the ancient Greeks on geometry. The most famous application of the method was that of Euclid's *Elements* (300 BC). The Euclidean system was an attempt to obtain all basic statements of geometry by pure derivation, based on a small set of 'axioms' whose truths were self-evident. The axioms were considered the 'first principles' known to be necessarily true, without need for justification. Until the nineteenth century, Euclidean geometry was conceived as being true as part of a physical theory of space.

It was not until a non-Euclidean geometry was discovered by Nikolai Lobachevskii (1792-1856) and Janos Bolyai (1802-1860) that the self-evident 'truth' of axioms was questioned.[10] Further work in non-traditional geometries stimulated analysis of the axiomatic method and created a 'foundational crisis' for some philosophers of mathematics.[11] With non-

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[10] Kline (1967) states that the existence of non-Euclidean geometries had profound implications for mathematics: "The most important effect of this creation has been the realization that mathematics doesn't offer truths. The Greeks adopted the axioms of Euclidean geometry because they believed that they were self-evident truths about physical space. The axioms appealed to their minds as necessary truths which anyone must grant, even without experience… The belief that mathematics offers truths was firmly held by every thinking being until the creation of non-Euclidean geometry. But if several geometries which contradict one another all fit physical space, then it becomes very obvious indeed, that all of these cannot be the truth… (pp. 471-472).

[11] An axiomatical 'mathematical foundations' was needed. Basic concepts were to be rendered more precise, more complex concepts reduced to simpler concepts, and the consistency, completeness, and interpretation of axiom systems were sought-after. Russell and Whitehead pursued this mission in *Principia Mathematica* (1903).

Euclidean geometries and contemporary axiomatizations of set theory (including the Zermelo-Fraenkel axiomatization) it seems that various sets of (independent) axioms can't all be self-evidently 'true.'[12]

A more modern (and widely accepted) characterization of a mathematical 'axiom' is that 'it is a proposition composed of undefined primitive terms and is not provable from other propositions (and axioms) within the formal system'. The role of an axiom (and its content) within a formal system is to characterize certain primitive (undefined) terms. In an axiomatic system, the undefined terms of an axiom do not have any definite meaning (other than from their occurrence in the axioms) and may be interpreted in any way that is consistent with a given set of axioms. Axioms can be assumed-true only under a consistent interpretation (or model) that gives meaning to a formal system. An axiomatic system consists in accepting without proof certain independent axioms (or postulates).[13] David Hilbert (1862-1943) was a proponent of characterizing axioms in this manner. He was concerned with the independence of axioms, and their relationships to the rest of a syntactic formal theory. Hilbert (1934) maintained that systems of mathematics are formal systems concerned only with the manipulation of symbols and sets of stipulated operations, without attention to the meaning of the symbols. Formal systems *may be interpreted* as a set of meaningless assertions. Axioms provide implicit definitions with a simultaneous characterization of a number of other terms in relation to each other. An axiomatic system is not (always) a system of statements about a subject matter, but a system of statements of a 'relational structure.'[14]

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[12] On a historical note, Ernst Zermelo (1908) states a self-evident, objective, and pragmatic defense of the ‘unproven’ assertion of ‘the axiom of choice’: “That this axiom, even though it was never formulated in textbook system, has frequently been used, and successfully at that, in the most diverse fields of mathematics, especially in set theory, by Dedekind, Cantor… and others is an indisputable fact… Such an extensive use of a principle can be explained only by its self-evidence, which, of course, must not be confused with its provability. No matter if this self-evidence is to a certain degree subjective- it is surely a necessary source of mathematical principle… But the question that can be objectively decided, whether the principle is necessary for science, I should now like to submit to judgment by presenting a number of elementary and fundamental theorems and problems that, in my opinion, could not be dealt with at all without the principle of choice” ('A New Proof of the Possibility of Well Ordering,' reprinted in van Heijenoort (1967, p. 187). Zermelo then goes on to list theorems that require the axiom of choice.

[13] Seemingly unaffected by mathematical history, Frege (1903) believed that axioms are objectively true. Godel similarly held a metaphorical belief that 'axioms force themselves upon us as being true' (1990, p. 268). The older definition of 'axiom' is still found in some dictionaries and mathematics texts: An 'axiom' is 'a proposition regarded as self-evidently true, without proof' (e.g., Eric Weisstein 2009).

[14] See Eduardo Giovanni and Georg Schiemer, “What are Implicit Definitions?” (2021) for history and analysis.

Many modern mathematicians are inclined to accept Hilbert's characterization of an axiom, with its emphasis upon a non-deducible set of axioms in formal deductive theories. Terms found in the axioms of modern mathematics and logic may be defined solely from their use in a system of axioms and do not have a meaning until they are given an interpretation.[15]

In modern geometry, 'point' and 'line' are not explicitly defined. Similarly, in set theory, the words 'set,' and 'element' (i.e., 'set membership,' 'belongs') are undefined terms. Paul Bernays (1888-1977), who was an assistant to Hilbert, summarizes the epistemic viewpoint towards axioms adopted here. When considering axioms involving the system of *objects* 'point,' 'line,' and 'plane' and the *relations* 'lie,' 'between,' and 'congruent' Bernays (1922) says:

> ...axioms are in no way judgments that can be said to be true or false; they have sense only in the context of the whole axiom system. And even the axiom system as a whole does not constitute the statement of a truth; rather, the logical structure of axiomatic geometry in Hilbert's sense, analogously to that of abstract group theory, is a purely hypothetical one. If there is anywhere in reality three systems of objects, as well as the determinate relations between the objects, such that the axioms of geometry hold of them (this means that by an appropriate assignment of names to the objects and relations, the axioms turn into true statements), then all theorems of geometry hold of these objects and relationships as well. Thus, the axiom system itself does not express something factual; rather, it presents only a possible form of a system of connections that must be investigated mathematically according to its internal properties (translated in Mancosu, 1998, p. 192).

The main idea is that mathematicians can construct formal languages consisting of consistent inference rules and axioms without concern whether the languages are 'true' or 'correct.' Lara Alcock (2014) states an 'axiom' is a statement that mathematicians *agree* to *treat* as true (p. 9).[16] The adoption of axioms (as *assumed-true, not literally true*) is based upon their role in a consistent formal theory and depends upon how a theoretician constructs the theory.[17]

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[15] J.R. Brown (2008) illustrates a simple set of *meaningless axioms* that can be *proven* to be *consistent*: (1) For any two lonks, at most one ponk zonks both. (2) For any two ponks, exactly one lonk zonks both. (3) There are at least three ponks which zonk each lonk. In this theory, ponk and zonk are undefined objects and zonks is a relation. This model is sometimes called the seven-point geometry (p. 74). But no complex theory can be easily proved consistent.

[16] Morris Schlick (1925) characterized an 'axiomatic system' as a system of truths created with the aid of implicit definitions that do not at any point rest on the ground of reality. "On the contrary, it floats freely, so to speak, and like the solar system bears within itself the guarantee of its own stability" (p. 37).

[17] Hilbert's idea of allowing symbols to remain undefined in axioms was a major break from the thought of Gottlob Frege, who believed that axioms should express objective truths, and that defined terms should have meanings that fix their denotations. Hilbert's formalism has support from mathematicians but little support from philosophers with realist ontology. See e.g., Shapiro (2000), pp. 140-171. Other critics of formalism point to Hilbert's failure to prove

Axioms are Prescriptions

Hilbert's and Bernays' formalist position recognizing the independence and syntactic relationship of axioms to other propositions within a model is consistent with the definition of an 'axiom' that I propose here:

> An **'axiom'** is an independent foundational prescriptive assertion that underlies a set of stipulative definitions; including the vocabulary, grammar-syntax, and inference rules that measure a specified domain. Axioms cannot be deduced from other sentences in a formal system. An axiom is typically (but not always) adopted if it helps map (or represent) the physical world (or linguistic discourse) in a fruitful way.

This definiens addressing the question 'what is an axiom?' is concerned with the *epistemic status* of an axiom, compared to its *syntactic relation* within a formal system. It is consistent that Peano's axioms, Euclid's postulates, and the fundamental axioms of logic (including bivalence, identity, contradiction, transitivity) are *prescriptive* (not literally true or false) as rules of measurement. The following definition of a 'prescription' is presented several times in this book:

> A '**prescription**' is an assertion that purports to express a stipulation (or rule) upon a practice, where its correctness (or incorrectness) is *dependent* upon its acceptance (or non-acceptance) by particular persons.

On the epistemic view here, axioms can only be *prescribed* as foundational rules for the (accurate) measurement of a given domain of interest.[18] This characterization of an axiom as being prescriptive is vastly different from philosophers and mathematicians who believe that axioms and 'laws of logic' are the result of an *a priori* or a self-evident investigation of reality.[19]

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the foundational consistency and completeness of mathematics (from Godel's theorems) as a reason for rejecting formalism. But Hilbert's failed search for these 'foundational' proofs doesn't undermine the tenets of his formalism. Foundational proofs aren't needed because *in fact* there are no true epistemic foundations for mathematical truths and certainty. That there are no true foundations for deductive systems is an implied conclusion of this essay.

[18] Less often do mathematicians engage in 'pure mathematics' and present a consistent set of definitions and axioms for a deductive system without having some heuristics and assumptions that these definitions are to be used to measure something. With pure mathematics, there is no reason to accept such assertions, except to follow their consequences as a meaningless deductive game. But some mathematicians have developed interesting axiomatic systems without practical interpretation that were later found to be very valuable and applicable to a domain.

[19] Compare to P.T. Geach (1976) who argues for and explains the 'self-evidence' of the bivalence axiom and its independence from empirical observation: "… we need no observation to show that either it is raining, or it isn't, nor to show that 3 is odd. Propositions like these, evident independently of observation (or testimony, or memory, for that matter), are called *self*-evident… *all* truths of logic either are themselves self-evident or follow by evident methods of proof from self-evident truths of logic…" (p. 71).

For instance, it is widely believed that the Law of Identity (i.e., for any object x, it is necessarily the case that x is identical with x) is a statement applicable to any object in any possible world; for it is inconceivable that an entity isn't identical with itself. On the contrary view held here, the identity axiom is understood as an implicit definition of how the word 'identity' is to be used in formal system. The law of identity in effect just *stipulates* that (numeric) 'identity' is a relation that each object bears to itself in every possible world. It is not a self-evident (or necessary) truth of reason. The major claim here is that *none* of the mathematical or metaphysical axioms, listed above, are literally true or false, much less 'necessary truths.'

The Difficulty of Complete Formal Axiomatization: The Post-Postulation of Axioms

In practice, axiomatic theories are useful when they can be used to model or engineer some aspect of the world. Theoretical work in empirical science most often consists of constructing or discovering mathematical models of physical phenomena. Eudoxus (408-355 BC) proposed the first formal axiomatic system based on Aristotle's statements involving axioms, postulates, definitions, and rules of inference. As discussed above, Euclid systematized the theorems of Eudoxus in the *Elements* (300 BC). It is typical that deductive systems start with an interpretation (i.e., a meaning) and associated definitions, and that axiom systems are constructed afterwards, based upon the concepts found in the definitions. It is presumed that a reasonably simple list of axioms can be interpreted from a deductive system. Howard Eves (1990, p. 13) describes the construction of an axiomatics:

> Now both the initial and derived statements of the discourse are about the technical matter of the discourse and hence involve special or technical terms. The meanings of these terms must be made clear to the reader, and so, the Greeks felt, the discourse should start with a list of the explanations and definitions of these technical terms. After these explanations and definitions have been given, the initial statements called *axioms* and/or *postulates* of the discourse are to be listed. These initial statements, according to the viewpoint held by some of the Greeks, should be so carefully chosen that their truths are quite acceptable to the reader in view of the explanations and definitions cited.

To repeat the above: The axiomatization of a domain typically starts with piecemeal concepts and definitions, which are *later* brought into a systematic formalization for understanding the overall deductive structure of a discipline. First come (useful) concepts and definitions and then come proposed axioms and recognition of undefined or implicitly defined (primitive) terms.

Putting together a concise set of axioms from an existing set of definitions is immensely difficult. As Gerard Venema (2002) notes, the axiomatization of all of mathematics has *not* been achieved. Even though it is sometimes thought that every branch of mathematics should be fully formalized as an axiomatic system, only with geometry has the axiomatic method been fairly successful and extensively used (p. 17). Peter Smith (2007) admits that in even the most tough-minded mathematics texts, the development of axiomatized theories are written with an informal mix of ordinary language and mathematical symbolism. Proofs are rarely spelled out in every formal detail, and so the presentation falls short of the logical ideal of full formalization:

> But we hope that nothing stands in the way of our more informally presented mathematical proofs being sharpened up into fully formalized ones- i.e., we hope that they *could* be set out in a strictly regimented formal language of the kind logicians describe, with absolutely every inferential move made fully explicit and checked as being in accord with some overtly acknowledged rule of inference, with all of the proofs ultimately starting from our explicitly given axioms. True, the extra effort of laying out everything in this kind of detail will almost never be worth the cost in time and ink. In mathematical practice we use enough formalization to convince ourselves that our results don't depend on illicit smuggled premises or dubious inference moves, and leave it at that... But still, it *is* absolutely essential for good mathematics to achieve precision and to avoid the use of unexamined inference rules or unacknowledged assumptions. So, putting together the logician's aim of perfect clarity and honest inference with the mathematician's project of regimenting a theory into a tidily axiomatized form, we can see the point of an *axiomatized formal theory* as a composite ideal... (pp. 18-19).

Of more importance than the difficulty of axiomatization, is Godel's (1931) proofs discussed below, showing no sophisticated formal axiomatized system is capable of proving every mathematical truth. Some theorems remain conjectures. Axiomatization has no real value.

Implicit Standard Assumptions Underlying Deductive Logic

Our attention is now turned to some of the implicit assumptions that underlie the practice of symbolic logic (and all of mathematics). We are in effect describing a 'meta-logic' because we are not talking about the formal structure of any single deductive system, but instead we describe with a short list, the beliefs, values, and definitions that lie behind most deductive systems. There are approximately fourteen assumptions that lie behind the formalization of any deductive logic (and are found in most introductory logic texts). For philosophers, these assumptions and definitions are rudimentary and are rarely questioned. But as indicated with a star*, the assumptions 2, 3, 10, 11, 12, 13, and 14 are all subject to dispute in the course of this book:

1. 'Logic' is the study of the methods and principles used to distinguish correct reasoning from incorrect reasoning. There exist similarities among arguments in natural language; and with a formal language a logician should represent those similarities and identify truth-conducive (i.e. truth-generating) argument forms (i.e., valid arguments).

*2. The Law of Bivalence: Every declarative sentence/statement expressing a proposition (in a domain) has exactly one truth value, either true or false.

*3 The Principle of Excluded Middle: A sentence/statement/proposition is either true or false, as a declarative sentence; in contrast to questions, exclamations, commands.

4. An 'argument' consists of premises and the conclusion. The 'premises' are a group of sentences that gives reasons/evidence/support for believing the conclusion. The 'conclusion' is the sentence that the premise(s) are claimed to imply (or entail).

5. An 'inductive argument' claims to support its conclusion with some degree of probability. For example: (1) 99% of the people in Springfield own a dog. (2) Mr. Brown is a resident of Springfield, so therefore (3) Mr. Brown probably owns a dog.

6. A 'deductive argument' claims that its premises support its conclusion necessarily. A deduction is relative to a particular vocabulary, syntax, and inference rules.

7. Three consistent definitions of 'validity': An argument is 'valid' if and only if it is necessary that *if* all its premises are true, its conclusion is true. An argument is 'valid' if and only if it is impossible for all the premises to be true while the conclusion is false. When an argument is valid, the premises 'entail' its conclusion.

8. An argument is 'sound' if and only if it is valid, and all its premises are true.

9. An 'inference' is when a *person* reaches or affirms a proposition on the basis of other propositions. An inference (with premises and conclusion) can be put into the form of an inductive, deductive, or abductive argument.

*10. There are three types of sentence/statement/proposition: A proposition is 'contingent' when it could have been true or false (e.g. Lincoln was elected president). A proposition is a 'tautology' (e.g., p or not-p) if it has substitution instances that could only be true (e.g. 'Either Lincoln was elected president or Lincoln was not elected president' has the form of a tautology and is thus a logical truth). A proposition is 'contradictory' (e.g., 'p and not-p') if it has substitution instances that are self-contradictory (e.g., 'Lincoln was elected president and Lincoln wasn't elected president' is false in virtue of its logical form).

*11. A sentence is contingently true, e.g., 'snow is white,' if and only if it is true that snow is white. Truth is the property of sentences in a possible (or the actual) world.

*12. The Principle of Semantic Reference: Words that are found in complete sentences and used in a context (a) refer to entities, (b) have meaning, and (c) are about something.

> *13. The Principle of Compositionality: Words are the basic components of sentences and the meaning of sentences depends (systematically) upon the meanings of the words that they are composed of.

> *14. For a speaker to know the meaning of an assertion, the speaker must be able to grasp its truth conditions (i.e., know what states of affairs constitute its truth condition).[20]

In advocating the descriptive-prescriptive distinction, the acceptability of the laws of bivalence, and excluded middle, for declarative sentences has been vigorously questioned. Whether 'truth' is a *semantic property* of sentences (in a model) or whether 'truth' is better conceived of as a *correspondence relationship* between a person's assertion of a *proposition* and *a state of affairs* is discussed in chapters ten and eleven. Assumptions #11 to #14 are discussed in chapters ten, eleven, and twelve. For now, we just observe that these stipulated definitions and assumptions are routinely adopted by practitioners of a truth-functional formal semantics.

The Status of Definitions in Formal Deductive Logic and Mathematics

What is the epistemic status of definitions found in logic and mathematics? This is a difficult question. It presupposes that we have an understanding of what a 'definition' is and what kind of definitions there are. In chapter six, a disjunctive tripartite definition of the concept of 'definition' was hypothesized: **x** is a '**definition**' in a definiendum-to-definiens relationship if and only if it is (1) reportive, or (2) theoretic, or (3) stipulative: (3a) an initial naming assertion, or (3b) an abbreviation, or (3c) is a precise formalization for practical, technical or personal reasons. It was contended that the pattern of definitions found in the formal deductive sciences are stipulations of the 3b or 3c variety (with mixed simultaneous 3a initial naming). Let us assume, against platonists, that mathematical definitions are not 'theoretic definitions' about real and eternally existing objective (mind-independent) structures. If mathematical definitions are obviously not reportive, and not theoretically objective, then most of them can only be stipulative in one of the two forms, 3b or 3c.

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[20] Michael Dummett (1978) says that a statement gets its meaning by being correlated with a state of affairs. That state-of-affairs is the statement's truth condition. The correlation between statement and truth condition is secured by (1) the referential relations that individual terms bear to objects in the world, and (2) by the way that they are combined into a sentence. To know the meaning of a statement is to grasp its truth condition (pp. 223-225).

Mathematical Definitions Are Stipulated Abbreviations (the 3b form)

Mathematician Morris Kline (1967) provides an intuitive view of how 3b abbreviatory definitions work in mathematics:

> Like other studies mathematics uses definitions. Whenever we have occasion to use a concept whose description requires a lengthy statement, we introduce a single word or phrase to replace the lengthy statement. For example, we may wish to talk about the figure which consists of three distinct points which do not lie on the same straight line and of the line segments joining these points. It is convenient to introduce the word triangle to represent this long description. Likewise, the word circle represents the set of all points which are at a fixed distance from a definite point. The definite point is called the center, and the fixed distance is called the radius. Definitions promote brevity (p. 51).

According to mathematicians James Robert Brown (2008) and John Horty (2007), this is the 'standard' or the 'official' view of mathematical definition. The standard view maintains that definitions are constructions that are neither true nor false. Definitions posit an abbreviation of a linguistic definiendum to a linguistic definiens. Definitions are stipulated for clarity and convenience. Most often the equal sign (=), the bi-conditional sign (iff), or a definition sign (df) are used to state that the linguistic sign on the left side (i.e., definiendum) is the same (or is identical) to the content on the right (i.e., the definiens). Philosophical (and mathematical) definitions are intended to be 'neutral' in content. Definitions should play no substantive role among the premises of a deductive argument and should play no role in the outcomes of deductive proofs and arguments. Bertrand Russell (1872-1970) and Alfred North Whitehead (1861-1947) in *Principia Mathematica* (2nd edition, 1903, p. 11) endorse this standard view:

> A definition is a declaration that a certain newly introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known. It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with symbols, not with what they symbolize. Moreover, it is not true or false, being an expression of volition, not a proposition.

This standard view is assumed by Patrick Suppes in *Introduction to Logic* (1957). In a chapter entitled 'A Theory of Definition,' he states that a respect for *definitions as abbreviations* is crucial when solving deductive proofs from an already specified set of adopted existents. Here are some examples of the use of 3b definitions, found in mathematics:

(1) The symbol **'v'** is to be used to designate 'not'

(2) A number *n* is **'even'** if and only if there exists an integer *k* such that $n = 2k$.

(3) A function *f* from the set X to the reals is **'bounded above'** on X if and only if there exists M in the reals such that for all x in X, f(x) is less than or equal to M.

One final notational note about 3b definitions. In many mathematical texts, these 3b definitions are presented as 'symbols' and 'meanings.' Here we will refer to 'definiendums' and 'definiens'.

Mathematical Definitions Are Formalizations (the 3c form)

It seems obvious that the 'standard' stipulative abbreviation view of definition, stated by Russell, Suppes, Brown, and Horty cannot be the whole story about mathematical definitions. Before abbreviations can be applied to existents, there must be a mode for how new terms and entities are introduced. Let's illustrate how mathematical definitions can be viewed as non-objective 'formalizations,' 'explications' or 'creative constructions' by considering a formal language such as Euclidean geometry.

The notions of point, line, and straight were natural language concepts before the work of early mathematicians. The following three sentences are examples of a 'precise formalization' of these natural language terms in the *Elements* of Euclidean geometry:

1. A 'point' is that which has no parts.
2. A 'line' is length without breadth.
3. A 'straight line' is a line that lies evenly with the points on itself.

Euclid asserted these precise definitions as part of an attempt to enunciate the smallest basic definitions that underlie the practice of geometry and arithmetic.

Euclid's definitions included geometric concepts that were used by previous geometers:

1. A 'boundary' is that which is the extremity of anything.
2. A 'figure' is that which is contained by any boundaries or boundary.
3. A 'circle' is a plane figure contained by one line such that all the straight lines falling upon it from one *particular point* among those lying within the figure are equal.
4. The *particular point* in definition 3, is called the 'center' of the circle.

From an epistemic point of view, it appears that the definitions of these seven geometric entities (point, line, etc.) as stated above are asserted as stipulative definitions in the 3c sense, as precise formal improvements to the preexisting ordinary language use of these concepts.

Intuitively these geometric concepts do not seem to represent independent natural objective theoretic entities. Instead, they are stipulated as a means for fruitful measurement and determining what extensions fit those specifications. These four definitions are not subject to being true (or false) but instead they are technical stipulations (explications or creative constructions) prescribed for acceptance and adoption.

On the view advocated here, the definition of a structured mathematical entity such as a 'triangle' should not be understood as referring to an eternal abstract existent (i.e., a form), but is instead the stipulation of an item of interest by the ancients long ago:

> 3) '**x** is a triangle' if and only if **x** lies in a plane, **x** is closed, **x** has exactly three sides, and **x** has straight sides.

This geometric entity, as well as others (e.g., parabola, ellipse, hyperbola), and the definitions of arithmetic (e.g., number, successor, addition), and those of formal logic (e.g., negation, conjunction, disjunction, conditional, and bi-conditional; defined by truth-tables) are all initially stipulative fixed definiens definitions of the 3c form.

Another important 3c form of fixed-definiens definitions found in mathematics is that of 'recursive definition.' A ‘recursive definition’ (also called 'inductive definition' and 'definition by recursion') is a definition in three clauses: (1) the expression defined is applied to certain items (the base clause); (2) a rule is given for reaching further items to which the expression applies (the recursive, or inductive clause); and (3) it is stated that the expression applies to nothing else (the closure clause). The characteristic features of a recursive definition: one or more clauses non-circularly define the most basic members of the set being defined, followed by one or more recursive clauses defining how other members of the set are built out of the more basic members. Below, the concept of being part of a 'family' is an example of a recursive definition:

(*x*) (*x* is in Smith's family = *x* is Smith,
    or *a* is in Smith's family
        And *x* is married to *a*
    or *a* is in Smith's family
        And *x* is born to *a*
    or *a* is in Smith's family
        And *x* is adopted by *a*).

A recursive method works when there is a finite number of types of basic members of the set and there are only a finite number of ways in which non-basic members can be built up or added.[21]

[21] In arithmetic, the recursive definitions of the ‘natural numbers’ can be derived from the basic principles of set theory. ‘0’ is stipulatively defined to be the empty set 0, ‘1’ is defined to be the set {0} (i.e., the set containing one member}, ‘2’ is defined to be the set {0, 1}, ‘3’ is defined to be {0, 1, 2}, and so on. All properties of the natural numbers can be proven using these recursive definitions and elementary set theory.

A third kind of a stipulated fixed-definiens definition found in mathematics, is that of a 'function.' A simple example of a function is where we define "function (m)" so that (m)x is the mother of the person x for all persons (who are elements of a set). If Jessica Alba is semantically designated as the person (as input x), the function specifies Catherine Alba as her mother. This well-defined function assumes that each element x is mapped to a unique element y (every person x has exactly one mother y). The output (extensions) of a function are designated by its input and fixed definiens. Alfred Tarski (1946) shared the following, stating that *definitions* are:

> …conventions stipulating what meaning is to be attributed to an expression which has thus far not occurred in a certain discipline, and which may not be immediately comprehensible…For this purpose it is necessary to define a symbol first, that is, to explain exactly its meaning in terms of expressions which are already known and whose meanings are beyond doubt… every definition may assume the form of an equivalence; the first member of that equivalence, the DEFINIENDUM, should be a short grammatically simple sentential function containing the constant to be defined; the second member, the DEFINIENS, may be a sentential function of an arbitrary structure, containing, however, only constants whose meaning either is immediately obvious or has been explained previously… In order to emphasize the conventional character of a definition and to distinguish it from other statements which have the form of an equivalence, it is expedient to prefix it by the words such as "*we say that*" (pp. 33- 35)

Tarski states that a new symbol when introduced as a mathematical definition must possess a meaningful definiens, before it can be committed to a theory. To define a symbol, we must first explain its meaning in terms of expressions already known. (That a definiendum should be a sentential function can be ignored). Tarski's statement that definitions are conventionally prefaced by "we say that" effectively covers both 3c formalizations and 3b abbreviations.

Although not explicitly concerned with the concept of 'definition,' Rudolf Carnap (1947) emphasized the importance of the 3c sense of definition, where sometimes the task of logical thinking is to replace a vague concept with a more precise (or technical) one. The clarification (or explication) of a concept should be expressed in a language framework that makes it precise and transparent in relation to other concepts. For Carnap there was no single correct language of measurement; multiple possible languages are possible. For Carnap acceptance of a 'formalized redefinition' of a concept cannot be judged true or false, but it is part of the acceptance of a language where using the term will be expedient or conducive, to the measurement of a domain. A dialogue is needed among practitioners to decide what formal system and what concepts work the best to measure a domain.

The Initial Defining of Fixed Definiens Concepts

Besides the 'formalization' and the abbreviation of terms already in use, theoreticians of any discipline must be able to construct definitions with new terms to designate entirely new and exotic distinctions. We can imagine that the concepts of 'prime' and 'vertex' were each: given a fixed definiens (3c), initially named (3a), and later abbreviated (3b) by ancient mathematicians during the construction of arithmetic and geometry:

> 1) A 'prime number' is a natural number greater than 1 that has no positive divisors other than 1 and itself.
>
> 2) The point at which two line-segments meet, is called a 'vertex.'

These two terms represent what may be called 'fixed definiens concepts' (i.e., 'closed concept,' 'formal concept'). These concepts have two characteristics that make up their uniqueness. First, a fixed definiens concept is a term that is stipulatively defined to *unequivocally identify* any item(s) that fall under its definition. The definiens is precise enough to distinctly exclude any entity that doesn't fall under the definition. Second, a fixed definiens concept is stable and not subject to alteration (without creating a new concept). The definiens determines what a term's proper referents (i.e., extensions) are, if any. The initial naming of a 'new concept' is the rarest form of definition in mathematics and in other academic disciplines, with the notable exception of philosophy, where terms and distinctions are introduced promiscuously at a philosopher's will.

Not All Fixed Definiens Concepts Are Defined by Initial Stipulation

Although fixed definiens concepts are always stipulatively defined to *unequivocally identify* any item(s) that fall under its definition, or to specify a fixed function, the definiens for an implied fixed definiens concept can be exceedingly difficult to consistently state. Carnap (1928) states that concept formation can be "intuitively projected and maintained, but there is no recognition what the thus formed concepts actually mean" (p. 306). For example, the concept of a "derivative" of a function was put to good use for nearly two centuries before it was given a precise definiens by the work of Cauchy and Weirstrass. In the history of mathematics, Carnap observes that there was difficulty in stating the precise fixed definiens definitions (of the 3c form) for 'derivative' and 'limit' which were initially conceptualized and formalized in a less precise and informal 3c form. A quote from Carnap:

> The inventors of the infinitesimal calculus (Leibniz and Newton) were able to answer questions concerning the derivative (the differential quotient) of common mathematical functions; for example, the derivative of the function x-cubed is the function 3x-squared. However, they could not say to what question this expression is an answer, that is, what is actually to be understood by the 'derivative' of a function. They could indicate various applications (for example the direction of the tangent) but they could not give a precise definition of the concept 'derivative.' To be sure, they believed that they knew what they meant by this expression, but they only had an intuitive notion, not a conceptual definition… However, their formulations for this definition used such expressions as "infinitesimally small magnitudes" and quotients of such, which, upon more precise analysis, turn out to be pseudo concepts (empty words). It took more than a century before an unobjectionable definition of the general concept of a limit and thus of a derivative was given. Only then all those mathematical results which long since been used in mathematics were given their actual meaning (1928, pp. 306-307).

This example illustrates that from an informal 3c stipulative definition of 'derivative', a more precise *explication* of a formal 3c definition evolved and was developed, where a 3b abbreviation for the term 'derivative' was adopted.[22] The concept of 'limit' and other mathematical symbols were used in explicating a precise 3c definition of 'derivative.' The derivative of *f*(x) with respect to x is the function *f*"(x), is technically defined as an explicit mathematical formula (found in contemporary textbooks, with exotic symbolization).

For fixed-definiens concepts, such as 'derivative' and 'limit' a precise definiens of a 3c form was sought that has a consistent relationship with other postulated fixed definiens concepts.

The Relationship Between 'Synonymy' and 3b & 3c Definitions

Synonymy may be defined as an existing relationship between terms, where there can be a substitutivity of terms (i.e., linguistic entities) in a formal or ordinary language context without altering the truth-value (or content) of the propositions in which the term occurs. In other words, a 'synonymy' is a descriptive relationship between linguistic entities (words, phrases) such that those linguistic entities are (in fact) substitutable within well-formed sentences where the intended propositional content of the sentence is retained. The terms 'oculist' and 'eye doctor' are

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[22] A.W. Carus (2007) states that *'explication'* in Carnap's view is the main task of conceptual engineering. It consists in the replacement of a vague concept (i.e., the explicandum) by a more precise one, the explicatum: "The first step is the clarification of the explicandum, the establishment of some basic agreement, among those using the vague concept, what they mean by it. The next step is a proposal for its replacement, a proposed explicatum, which should have the most important uses that were agreed on in the clarification stage, but need not have all of them. It should also, if possible, be expressed in a language framework that makes precise and transparent its relations to other concepts (p. 40).

synonymous, as are 'bachelor' and 'unmarried male' synonymous. Synonyms are mutually substitutable in all contexts without change, no matter whether one is personally familiar with the terms involved.[23] When recognizing synonymies as intersubstitutable linguistic forms, Alan Cruse (2011) says "this is a very severe requirement, and few pairs, if any, qualify… Absolute synonyms are vanishingly rare, and do not form a significant feature of natural vocabularies" (pp. 142-143). John Lyons (1995) says that "absolute synonymy is extremely rare" (p. 61). John Saeed (2009) concurs that "exact synonyms are very rare" giving couch-sofa, lawyer-attorney, and large-big, as examples (p. 65).[24]

What is the relationship between 3c formalized definitions, 3b abbreviatory definitions, and synonymy? The answer is illustrated with the evolving development of the concept of 'derivative' above. First, an inexact and imprecise, but intuitive, formalized concept of 'derivative' is recognized as a 3c definition. With much research, the transition of a 3c stipulative definition is made more precise into a very complex mathematical formula.

This non-arbitrary definiendum-to-definiens formula then is transitioned into a formally arbitrary 3b equivalence using 'derivative' (or symbol) as the shorter definiendum term and the *algebraic formula* its definiens. With a 3b definition, 'derivative' maintains its traditional (pre-analyzed) use in mathematics, and it formalized definition fulfills a standard of unambiguous eliminability.[25] Although a 3b abbreviation is *not* an assertion of an *existing* synonymy, the relation between terms (i.e., definiendum and definiens) in a 3b definition may be fruitfully *'treated as' or 'assumed-synonymous'* within a mathematical theory, or formal argument. This pattern of notational equivalencies supports an anti-realist ontology of mathematical entities.[26]

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[23] For example, few people know that the terms 'furze' and 'gorse' are synonymous, unless they are familiar with these terms' meaningful definiens: a spiny yellow-flowered evergreen shrub of the legume family.

[24] The definition of 'synonymy' as 'content-preserving linguistic substitutivity' differs from the (less-precise) definition found in most dictionaries, which states that 'synonymy' is a relationship of words in the same language that have the *same* or *very nearly* the same *meanin*g. *Roget's Thesaurus* contains these synonymies. In ordinary language 'synonymies' are often exaggerated (e.g., 'Disney World is synonymous with crowds,' 'The first Saturday in May is synonymous with the Kentucky Derby,' 'Rikers Island Jail is synonymous with violence and neglect.').

[25] Quine (1936) apparently describes the nature of 3b definitions saying that "Functionally a definition is not a premise to a theory, but a license for rewriting a theory by positing definiens for definiendum or vice versa. By allowing such replacements a definition transmits truth; it allows statements to be translated into new statements which are true by the same token (p. 330).

[26] The existence of a mathematical entity is relative to a structure that consistently satisfies the axioms of a formal model. The consistency of an axiomatized formal theory shows that an axiomatization *could* have instantiated models but not that it actually does.

Mathematical Concepts and Formalized 3c Definitions

It has so far been shown how 3b and 3c definitions are instrumental for formulating mathematical theories and proofs. While 3b definitions are explicitly recognized, do mathematicians recognize the existence of 3c definitions, and if so, how? Given elementary expositions about the structure of mathematics, it is apparent that mathematicians often speak of 3c definitions in terms of the analysis (or definition) of *concepts* and *objects*. A definition is said to define a single (fixed definiens) concept. A definition may precisely specify what a certain function is, for example: a number n is **'even'** if and only if there exists an integer k such that $n = 2k$. The concept of 'even' is said to *apply* to n if and only if n satisfies (is true of) the definiens (i.e. an integer that can be doubled from another integer). Certain objects (i.e., integers) are said to satisfy the properties of the mathematical concept of **even**. Suppose that n is 12. Since there does exist an integer 6 (as a variable to k) where 2(6) is 12, this shows that 12 satisfies the definition of 'even.' This illustration shows how formalized 3c definitions are often understood as the stipulation and analysis of fixed definiens concepts. The explication of mathematical concepts, their relations, and objects are a fundamental part of mathematics.[27]

Answer to Question #1: Axioms, Definitions, and Inference Rules Are Prescriptions

We have arrived at a general conclusion to the first questions posed above: What is the epistemic status of axioms, definitions, and inference rules in mathematics? Can we know them to be true? Our conclusion is that the *introduction of a vocabulary, the syntactical formation rules, inference rules, axioms*, and *definitions* of mathematics (and deductive systems in general) are *prescriptive* and are not knowable as true. The 'axioms' of a formal system are a framework of relationships among primitive terms that are determined only after the basic concepts and definitions of a measurement system have already been specified.

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[27] Kline (1967) states that viewed as a whole, mathematics has a structure of a collection of branches: "The largest branch is that which builds on the ordinary whole number, fractions, and irrational numbers, or what, collectively, is called the real number system. Arithmetic, algebra, the study of functions, the calculus, differential equations, and various other subjects which follow the calculus in logical order are all developments of the real number system…A second branch is Euclidean geometry. Projective geometry and each of the several non-Euclidean geometries are branches as are various other arithmetics and their algebras… with many more divisions" (p. 541).

Do Logical and Mathematical Entities 'Exist'?

Having completed our observations about the prescriptive elements found in the construction of symbolic deductive systems, we now turn to existence claims. What is the status of the 'existence' of entities that are introduced as part of a logical discourse? Do mathematical concepts (e.g., point, circle, number, addition, identity, prime number, ratio) exist as objective entities discovered by the logician/mathematician, or are the definiens of these concepts invented (i.e., stipulated) as a means for the measurement of various domains? How can we have knowledge about logical-mathematical entities?[28]

The answers to these questions have already been indicated. In an interpreted formal system, logical-mathematical entities have no independent objective existence, but are stipulated to exist (i.e., invented) using definitions with a fixed definiens. Speakers do not *refer* to points, circles, numbers, and ratios as existing *objective* unified entities when talking about them; instead, speakers *use* these mathematical terms consistent with stipulated definitions.[29] The position advocated here is termed 'anti-realism' (in ontology) because it denies the existence of independent objective mathematical entities.[30] Mathematical entities exist as *stipulated* entities

[28] In recent history, there have been two dominant opposing viewpoints concerning these questions; realism and nominalism. Platonic realism and nominalism ask the same questions: (1) What is mathematics about? (2) What are mathematical objects? and (3) Are there real mathematical objects (i.e., Do mathematical objects exist?). Platonic realists believe that mathematics is about real objects of which we can have *a priori* knowledge. Nominalism states that mathematics cannot be about real objects, because mathematical objects don't exist. Putnam (1975) has stated "the traditional debate between Nominalists and Realists, whatever its status, is irrelevant to the philosophy of mathematics" (p.40). I agree and advocate a conceptualist position (popular in the 19th century) in chapter thirteen.

[29] In chapter ten, I argue it is false that *linguistic entities* (including mathematical items and sentences) in a context (a) possess meaning, and (b) can be about (or refer to) something. In contrast, *speakers* refer to entities, and the meaning of a sentence (and what it is about) depends on the intent and use of the sentence in a context by a speaker.

[30] Ontological 'mathematical realism' contends that mathematics is about the discoverable objective features of the world. For example, it is claimed that numerals denote objective numbers, and that numbers are abstract and eternal. The realist position has its roots in the philosophies of Plato and Aristotle and is still espoused by a significant number of philosophers today. Plato thought that mathematics was about a causally inert realm of abstract objects (numbers, circles, etc.) that are eternal and indestructible and are of an ideal unchanging realm of 'forms.' Aristotle believed that mathematical concepts were abstracted from, and discoverable as 'existing' in perceptible objects. Aristotle believed that abstract properties studied by mathematicians are inherited from the sensible particulars. Both Plato and Aristotle believed that the definitions sought for, and studied by the mathematician, represented something independent and objective, apart from the mathematician. A perennial problem with this realist view of abstract mathematical objects, is the question of how do we have knowledge about these nonphysical objective abstract objects? Both Plato's realm of objective forms and Aristotle's method of 'abstraction' of properties from sensible objects (e.g., 'surfaces' from physical bodies, and '5' from groups of five things) meet with epistemological difficulties. Contemporary logical-mathematical realists have serious problems in describing a plausible epistemology for their views. This is discussed in chapter thirteen.

that are postulated in a consistent and fruitful formal deductive system based upon (metaphysical) axioms; and definitions that have a precise fixed (unambiguous) definiens. Mathematical concepts are *creative constructions* used for (possibly) modeling something. Mathematical objects are not entities that exist outside of space and time that are acausal and eternal. This view is consistent with that of logician, Alonzo Church (1932), who states that "The entities of formal logic are abstractions, *invented* because of their use in describing the facts of experience or observation, and their properties, determined in rough outline by this intended use depend on their exact character on the arbitrary choice of the inventor" (p. 352).

Rejecting the 'Indispensability Argument' for the Existence of Mathematical Entities

The status of the concept of 'existence' or 'being' has been a traditional problem of metaphysics, and the problem of ontology (i.e., 'what exists') continues to be a subject of debate. With the works of Quine (1948, 1953) and Putnam (1975), it was claimed that what 'exists' in particular domains (or theories) depends upon whether there is a need for those entities in the successful scientific explanation, description, and prediction of natural phenomena. Quine thought that if we put our theories into a standard notation (e.g., first-order logic) then our 'ontological commitments' are revealed. For Quine, questions of ontology are about existential quantification: what objects are in the range of theory's variables? If certain theoretical physical entities and mathematical entities are indispensable for explanation, then Quine says that they 'exist.' According to this thesis, medium-sized physical objects, planets, electrons, and numbers *all exist* in the *same sense* because they are indispensible for scientific explanation. Supposed entities that are not needed (and we are not committed) to help explain the world, don't exist.[31]

The Quine-Putnam 'indispensability argument' (and 'ought' assertion) suggesting that *'existence' has a single sense* (even though a plurality of existents in various scientific domains is allowed) should be denied. As an alternative thesis, when it said that 'people exist' and 'numbers exist,' and 'fictional characters exist,' we are using *different senses of 'exist'* and that we are (tacitly) aware of different categories of existents and senses of exist. We assume that 'people

[31] Quine's argument affirming mathematical entities is as follows: (1) We ought to have ontological commitments to all and only the entities that are indispensible to our best scientific theories, (2) Mathematical entities are indispensible to our best scientific theories, and therefore (3) We ought to have ontological commitment to mathematical entities. The *first premise* and the *conclusion* can be understood as *prescriptions*. The first premise is debatable. The assumptions about personal justification and the epistemology of holistic theory justification are questionable. The indispensability argument is controversial and has been rejected by most philosophers.

exist' as concrete entities in a physical world, 'Sherlock Holmes exists' as a fictional character, and 'numbers exist' relative to a formal axiomatic deductive language system. Answers to any existence question comes from conceptual competence and background beliefs.

The Different Kinds of Existents and their Quantification

'Existence' is a group resemblance concept and there are different varieties of existents. In chapter six (and chapter nine) it is hypothesized that there are six primary kinds of concepts that help demarcate the kinds of basic existents (e.g., natural kind; group resemblance; fixed definiens, including mathematical objects; fictional; definite descriptions; and proper names). So, it seems plausible to believe that philosophical existence and quantification statements across 'objects' or 'somethings,' should include a specified (or presupposed) domain relative to these six basic existents. In ordinary language, natural kind (e.g., water), group resemblance (e.g., game), mathematical (e.g., triangle), and fictional objects (e.g., Spiderman) are considered existents.

When intelligibly asserting that something **x** exists, it is clear that the object **x**, must be qualified to a domain of discussion. For example, it is true that unicorns *do not exist* as natural kind objects, but they *do exist* as fictional objects. Amie Thomasson (2009) agrees with this kind of assessment as she states "… existence claims formulated quantificationally are complete and truth-evaluable provided a domain is properly specified, where that involves specifying (or at least tacitly presupposing) what sort or sorts of entity we are talking about" (pp. 464-465).

In contrast to Quine's ontological commitments as determined by the variables used in the logical notation of various sciences, and the metaphysical realists' presumption of a neutral sense of 'thing' or 'object' as a part of reality; the view of existents as advocated here, is that 'existents' are postulated relative to a domain of discourse (and involve the six primary kinds of concepts).

Answer to Question #2: Mathematical Entities Exist as Stipulated Constructions

We now respond to the second question above: Do mathematical entities 'exist' and if so, how? Mathematical objects are the consequence of stipulative definitions constructed explicitly, or are implicit within axioms, in a formal deductive system. An 'anti-realist' position is adopted here where the existence of mathematical entities isn't denied; it is just denied that mathematical entities 'exist' in an equivalent sense to the existence of objective concrete physical entities, group resemblance entities, or fictional entities.[32] Until a structure is shown to consistently satisfy the axioms of a formal model, its existents are hypothetical. The consistency of a formal theory will show that an axiomatization *could* have instantiated models (or domains) but not that it actually does. The generic question of 'what exists' doesn't have an absolute true answer.[33]

Mathematical Knowledge: A Description of Game Formalism

Mathematical knowledge is similar to knowing the rules of a game and making moves that accord within the rules of the game. If one adopts certain rules, then there are certain valid conclusions or outputs that follow, given certain inputs. This simple epistemic-semantic view of entailed mathematical truths is similar to what has been called 'syntactic formalism,' 'deductivism,' or 'game formalism.' Besides stating that the axioms of a deductive system express implicit definitions (and primitive terms) independent of any derivation from other propositions, formalism holds that deduced mathematical 'truths' are the consequence of following a consistent set of manipulation rules in a formal system. Reasoning proceeds based upon syntactically marked regularities of expressions without an immediate concern for semantics. The content of mathematics is exhausted by the rules operating within its language. The adoption of certain concepts, definitions, and rules are typically guided by the pragmatics of measuring a given domain (e.g., numerical, spatial, valid arguments). Propositions entailed from a proof are derived relative to a system's foundations (axioms, definitions, inference rules, grammar, and vocabulary).[34] We adopt this basic epistemology, and label it 'game formalism.'[35]

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[32] Mathematical fictionalism as endorsed by Harty Field (1980), Mark Balaguer (1998) and Mary Leng (2010) is denied. Mathematical fictionalism is a version of mathematical nominalism where it is maintained that there are no mathematical objects. It is the position that mathematical sentences and theories are not about abstract mathematical objects (as realists suppose) and so our mathematical theories cannot be true. For example, a sentence such as '3 is a prime number' is false because there is no real object '3.' Fictionalism is an error theory about mathematical discourse. It states that although we normally take many mathematical statements to be true, we do so mistakenly.

[33] Rudolf Carnap (1956) is an adherent of ontological anti-realism. He argues that whether it is theoretically useful to employ a given linguistic framework is settled largely on pragmatic grounds. The external question of whether there really is a realm of entities corresponding to certain linguistic frameworks is a pseudo-question since it assumes that the question is meant to be answerable independently of any internal linguistic framework. There is no way of confirming or disconfirming the reality of objects independent of a given linguistic framework. What kind of representational framework is best suited for modeling and clarifying substantive issues is a matter of linguistic 'explication.' Explication is thought of as the motivated stipulation of meanings, the setting up of frameworks with a clear semantics and well-defined rules in which the internal mathematical and empirical questions could be asked and answered. Carnap believed that questions about whether ordinary physical entities exist, or whether numbers exist, are *neither true nor false*, because they are external claims outside linguistic frameworks.

The incompleteness theorems of Kurt Godel (1931) that demonstrate that no (sufficiently powerful) recursive axiomatic system can be proved both 'complete' and 'consistent' does not affect the game formalism thesis. We can acknowledge that some well-formed formulas are not provable within a formal system and that some theorems in axiomatic systems remain as conjectures (i.e., as unprovable beliefs). That every truth-relative-to-a-formal system is not provable implies that a mathematical truth cannot be exclusively identified as it being the conclusion of a valid proof. Since there are more mathematical truths (derivable well-formed formulas) than what the axioms and definitions of a formal system can prove; axiomatic systems are incomplete. In addition, since there are unproven putative truths such as e.g. Goldbach's conjecture ('every even integer *n* greater than two is the sum of two primes') and e.g. the Continuum Hypothesis ('the cardinal number of real numbers is the next greatest after the cardinal number of the natural numbers') we must admit there are mathematical propositions ('conjectures') that are believed true, that haven't been, and perhaps cannot be proven to be true. A 'conjecture' is a putatively true theorem that hasn't been derived by formal mathematical proof. *Most entailed mathematical truths are deducible* as 'true-in-a-language' but *not all*.

Godel's second major conclusion that there can be no proof of the 'consistency' of a sufficiently sophisticated deductive system is no more troublesome than is Hume's problem of induction (chapter four). Godel shows that it is impossible to prove that a sophisticated formal system won't generate a contradiction. The natural reaction to Godel's theorem is that, if no inconsistency has been found in a given formal system, and no inconsistency is expected, and if the system is fruitful under an interpretation, then it should be adopted. Haskell Curry (1954) officially adopts this pragmatic position. We may recognize Hume's problem of induction as a 'peculiar fact' about induction being 'unjustifiable' in a sense, and likewise, Godel's theorems are true but peculiar facts about 'completeness' and 'consistency' in deductive systems.

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[34] A proof system is formed from a set of rules chained together to form proofs or derivations. Formal proofs are sequences of well-formed formulas (wffs). For a well-formed formula to be part of a proof, it might be an axiom or the product of applying an inference rule on previous wffs in the proof sequence. A symbol or a string of symbols comprise a wff if the formulation is consistent with the formation rules of the language.

[35] Mathematical realists and practitioners of a standard formal semantics have very strong objections to game formalism. Shapiro (2000, p. 145) says that "we use language to talk *about* things, usually things other than language itself. In its normal usage, a symbol *symbolizes something.*" Since the definiens of stipulated entities are specified linguistically and are not about an independent objective entity (e.g., object, concept, function), the realist finds it problematic about how numbers can exist and how mathematical knowledge can be *about* 'something.' For Shapiro, mathematical language has 'meaning' and game formalists ignore this meaning (p.170).

While L.E.J. Brouwer (1881-1966) used the term 'game formalism' as defamatory towards Hilbert's formalism, in contrast, the term 'game formalism' is welcomed here. 'Game formalism' is defined as 'the basic tenants of syntactic formalism (described above) without any requirement for rigorous proofs about consistency and completeness.'[36] The term 'game' has connotations of something being 'fun' or being a 'project,' which coincidentally suggests that attempts at the complete axiomatization of mathematical domains isn't important for practicing mathematicians with ordinary concerns. An extreme axiomatization may solve some 'intellectual puzzles' about a formal theory's structure; but has no practical consequence. Most mathematicians are currently satisfied that the axioms of set theory and model theory are the most basic background foundations of mathematics.

Logical Consequence

Logical consequence is the chief subject matter of deductive systems, including logic and mathematics. Logical consequence is a relation among claims (sentences, statements, propositions) expressed in a language. An account of logical consequence is an account of 'what follows from what.' Logical consequence (and validity) yields a way of evaluating the connections between a series of claims, or more specifically, of evaluating *arguments*. Logical consequence and validity are closely related: An argument is valid if and only if its conclusion is a logical consequence of its premises.[37] On the proof-centered syntactic approach to logical consequence, the validity of an argument amounts to there being a proof of the conclusions from its premises.[38] Whether or not a conclusion follows logically from some premises depends

[36] Oystein Linnebo (2017) describes 'formalism' as seeking to banish semantic notions from mathematics or else reduce such notions to purely syntactic ones (p. 44). The formalism described here just says that once a formal measurement system is devised (invented, constructed) for a domain, then its structure can be studied for consistency and consequences without concern for semantics. Formalism doesn't seek to 'banish' or 'reduce' mathematical semantics to syntax. Whether a sophisticated formal structure has any realizations is contingent.

[37] A formal argument is a sequence of well-formed formulas. Logical consequence and validity can be defined *syntactically* and *semantically*, i.e., in terms of the axioms or rules of the syntactic system, or in terms of its interpretation. 'Syntactic validity' is where a series of premises and a conclusion are valid-in-a-formal-language, just in case the conclusion is derivable from those premises and the axioms of a language (if any) and by the rules of inference in the language. 'Semantic validity' is where a series of premises and a conclusion are valid-in-a-formal-system just in case the conclusion is true in all interpretations in which those premises are true.

[38] One kind of 'proof' within formal deductive systems is that of proving theorems. A theorem is a basic formula of sentential logical equivalence that is necessarily true relative to and entailed by the axioms, syntax, and inference rules of a formal language, where its proof proceeds by suppositions as premises.

solely on the 'form' of the argument, and not the content. Proofs can be understood as instruction manuals about how to derive the well-formed formula we are trying to prove (i.e., the conclusion) from the well-formed formulas we have as premises by repeatedly applying the rules of inference. Following these rules, given true premises, any conclusion we derive through application of these rules of inference, will also be true. Any argument with a conclusion that can be derived from its premises is valid. Truth preservation is valued, which is the property of an argument in which the conclusion is never false when the premises are true.

There have been recent debates about whether there is one sense of 'logical consequence' or whether there are many (e.g., Haack 1978, Beall & Rustall, 2006, Shapiro, 2014). Logical monism is the view that there is one true logic with a single extension that holds in all rational discourse. Logical pluralism maintains that there is more than one correct system of logic. Without adding any argument, the pluralist response is favored here. Logical consequence is understood relative to a theory or structure and is made precise in more than one way. There is no single logical system that is correct (or best) in all contexts. Some formal representations may be better absolutely, or for certain purposes. There is no unique, ideally perspicuous formal notation in which the unique logical form of every informal argument is correctly represented. Any consistent axiomatization is worthy of some mathematical study.

But this isn't to claim that 'logical consequence' is a group resemblance concept. There is a single, intuitive notion of logical consequence. Some informal arguments are judged to be valid, others invalid. A formal logic is constructed in which the relevant structural features are schematically represented. Formal logical systems are an attempt to formalize informal arguments, to represent them in precise, rigorous, and generalizable terms. The formal argument should be valid in a system just in case the informal argument is valid in the extra-systematic sense. There is an extra-systematic concept of 'validity' to which logicians aim to give precise expression. A logical system is correct (i.e., acceptable, adopted) if the formal arguments which are valid in that system correspond to informal arguments which are valid in the extra-systematic sense, and the well-formed formulas which are logically true in the system correspond to statements that are logically true in the extra-systematic sense. In sum, an argument is either valid or not valid, relative to a well-formed deductive system. There is one relation of deductive consequence, and different formal systems do a better or worse job of modelling that relation.

How Can We Know '141678 + 639465 = 781143'?

We are now in position to state how deducible mathematical assertions such as 141678 + 639465 = 781143 (and e.g., 7 + 5 = 12) can be *known* where such an assertion is part of a consistent formal system. Under the PE definition of knowledge stated in chapter one, **S** knows 141678 + 639465 = 781143, if: (1) this proposition is *true-in-a-language* as being derived from the definitions and inference rules of arithmetic, and (2) if **S** *believes* this proposition to be true, and (3) if **S** *correctly performs* the addition operation to obtain the sum of 781143 (having *relevant reasons* for believing this sum), and if (4) **S** has *no doubt* of her computations (satisfying condition 4a). The conditions for knowing a mathematical proposition are subject to the four requirements of the PE definition. The knowledge of a logical consequence occurs when a practitioner is familiar with the rules of calculation, and then those rules are applied in a manner that is relevant (and without error) for why a deduced conclusion should be believed.

Under game formalism and the PE definition, 'mathematical knowledge' is treated as a *species* of knowledge, and *not a separate kind* of knowledge. This contrasts to Hilbert and Bernays who believed that mathematics is a *separate* true-in-a-language discipline where rigorous proofs show whether **p** is true-in-a-formal-language, **p** is impossible-in-a-formal-language, or **p** is unprovable-in-a-formal-language. While it can be granted that there are formal true-in-a-language calculations (e.g., 141678 + 639465 = 781143) and true-in-a-formal-language sentences (e.g., valid entailments, tautologies), there are also ordinary mathematical propositions that can be objectively true or false when applied to practical questions. For example, if Sam has 141678 marbles and Susie has 639465, then they (objectively) have a total of 781143 marbles. This proposition about a quantity of marbles expresses a state of affairs independent of acceptance (or acknowledgement) by persons.

The Objectivity of Mathematical Propositions

While derivable mathematical propositions can be known as true-in-a-language as entailed truths within a formal system; as just mentioned, these same propositions are also said to be 'descriptions' and 'objectively true' *when applied to practical questions* (e.g., the sum of Sam and Susie's marbles, above). For further examples, consider these case problems that might be presented to a high school math student:

1) Edison High School has 840 students, and the ratio of students taking Spanish to the number not taking Spanish is 4:3. How many of the students take Spanish?
Choice of answers: a) 280 b) 360 c) 480 d) 560 e) 630

2) Let the lengths of the sides of a triangle be represented by x + 3, 2x - 3, and 3x - 5. If the perimeter of the triangle is 25, what is the length of the shortest side? Choice of answers: a) 5 b) 6 c) 7 d) 8 e) 10

3) Sam has 141678 marbles. Susie has 639465 marbles. They decide to combine their collections of marbles to sell on the internet. If they combine their inventories, how many total marbles do they have in their starting inventory? Choice of answers:
a) 497787 b) 781143 c) 793033 d) 805213 e) 816743

It is usually thought (especially by high school mathematics teachers) that there is a *single objectively true answer* to each of these hypothetical problems, and that the answer is *knowable* to a high school student if the correct deductive reasoning is used. The teacher's belief is true. The belief that student **S** can know the objective truth of a mathematical answer (requiring deductive reasoning) is compatible with the fact that the correct answer to each of the above problems is 'descriptive' and is 'objectively true' under the following definitions:

> A '**description**' is an assertion that purports to express a correspondence (or a representation) of some state of affairs, where its correctness (or incorrectness) is *independent* of its acceptance (or non-acceptance) by particular persons.
>
> A description is **objectively true** if it expresses a correspondence (or a representation) to some state of affairs that is independent of its acceptance (or acknowledgment) by particular persons. A description is **objectively false** if it doesn't correspond to; or represent a state of affairs.

In the first example, if **S** believes that this problem can be solved by using the formula 4x + 3x = 840, and solving for x which designates a number of students, and multiplying by 4 to get answer of 480, then **S** has used the proper methodology and has relevant reasons for believing answer choice c. The result is that 480 is the objectively correct answer to the posed question in the situation specified. The answer is derivable as true-in-arithmetic, and its truth is independent of whether any student believes it to be correct. The answer of 480 is objectively correct because its truth is a reflection of the rules, concepts, and material conditions involved in the example. Similarly, with respect to the second geometry problem, if a student uses the equation (x + 3) + (2x - 3) + (3x - 5) = 25 to deduce that the sides are 8, 7, and 10, and that 7 is the correct answer to this problem, then again, the capable and confident student believes an objectively true answer based upon relevant reasoning. With the third example, if a student performs the addition function correctly by self-calculation, or by correctly entering the information into a reliable calculator, and transcribing its result, then b is known as the correct answer.

Answer to Question #3: What is the source of mathematical truth?

As promised in the initial paragraphs we can now put together an elementary explanation of where 'mathematical truth' fits in among our ordinary thoughts. In childhood, we gain epistemic access to mathematical concepts (number, ratio, circle, point, parallel, tangent) in a similar way to how we come to understand other fixed definiens concepts (e.g., equator, bachelor, circle, not, over, under, I). Our vocabulary is learned from interactions within linguistic communities. Typically, we learn terms and concepts and use them fluently in sentences without recourse to dictionary definition or a request for reportive definition.

With respect to mathematics, learning the language is more formal because we must learn the stipulations of definitions, functional connectives, and formulas by rote memorization in elementary school. Mathematical knowledge is similar to knowing the rules of a game and making the moves that accord with the rules of the game. Repetitive practice using the inference rules, grammar, and definitions of mathematics is essential for solving mathematical problems.

The true propositions of arithmetic, geometry, and symbolic logic are derived relative to the stipulated foundations (definitions, inference rules, grammar, vocabulary, and axioms) of the deductive system.[39] The introduction of logical and mathematical entities is accomplished by fixed definiens stipulative definitions (including those fixed definiens concepts about the ‘relations’ between entities).[40] From an initially well-formed language of prescriptions, we are able to calculate well-formed derivations that can be described as true-in-a-language (or a ‘logical consequence’ or 'entailed') and as objectively true when applied to certain real-world questions. 'Game formalism' best explains how we can come to know mathematical truths.

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[39] But, to repeat, as Godel's theorems demonstrate, not all deduced mathematical truths are formally provable from the rules of a formal deductive system. As a result, we *cannot* simply *identify all* mathematical truths as being the result of a derivation within a formal system.

[40] Michael Dummett (2010) affirms that fixed definiens concepts are sometimes introduced into mathematical discourse: "Mathematicians have sometimes to engage in conceptual analysis, seeking definitions of concepts such as numerical equivalence, continuity, and dimension. But their aims differ from philosophers. They care little whether the definitions that they arrive at capture the concept as we implicitly understand it in ordinary life: they are concerned only to formulate a precise concept under which it may be reasonably claimed that every case determinately either falls or doesn't fall. Having done so, their argumentation will proceed within the boundaries of the definitions that they have adopted" (pp. 10-11).

Towards a More Technical and Rigorous Explanation of Game Formalism

As stated earlier, this chapter aspires to new thinking about the nature of mathematics and logic, with a hope to inspire more rigorous and detailed analyses from philosophers with greater expertise. In his "Our Knowledge of Mathematical Objects" (2005), Kit Fine sketches what could be an initial starting point for the view presented here. Fine seeks a "new approach" to the philosophy of mathematics that he calls 'proceduralism' or 'procedural postulationism' which is similar to that as advocated by Hilbert (1930) and Poincare (1952). This involves the belief that the existence of mathematical objects and the truth of mathematical propositions are to be seen as the product of 'postulation.' He continues as follows:

> …But it takes a very different view of what postulation is. For it takes the postulates from which mathematics is derived to be imperatival, rather than indicative, in form; what are postulated are not propositions true in a given mathematical domain, but procedures for the construction of that domain (p. 89).
>
> Under standard forms of postulationism, what is postulated is the truth of a proposition. Thus, it is something that might be expressed by the means of an indicative sentence, such as 'every number has a successor.' A mathematical theory is then given a suitable set of indicative sentences or 'axioms.' Under our approach, by contrast, what is postulated, or prescribed, is a procedure for the construction of the domain. These procedures are more appropriately signified, not by other indicative sentences, but by imperatives or 'rules'; and a mathematical theory is to be given by a suitable set of rules for the construction of its domain.
>
> We might compare the rules, as so conceived, to computer programs. Just as a computer program prescribes a set of instructions that govern the state of a machine, so a postulational rule, for us, will prescribe a set of instructions that govern the composition of the mathematical domain; and just as the instructions specified by a computer program will tell us how to go from one state of the machine to another, so the instructions specified by a rule will tell us how to go from one 'state' or composition of the mathematical domain to another (one that, in fact, is always an expansion of the initial state)… (pp. 90-91).

In his twenty-page essay, Fine presents a brief sketch of an underlying technical program as a way to answer how we can acquire knowledge of mathematical objects. The details of Fine's analysis do not concern us here, but the overall viewpoint about what mathematics is, is similar in spirit to the worldview presented here.

Conclusion

In this chapter, we have tied the epistemology of mathematical entities to mind-dependent proposals involving the use of three kinds of stipulative definition (not always exclusive) and the implicit definitions implied in the specification of a deductive system's axioms. The ontology and epistemology of mathematics has been explained with an analysis of the initial prescriptive propositional structure of a deductive system as found in introductory logic textbooks. The introduction of logical and mathematical entities is exclusively accomplished by the introduction of fixed definiens stipulative definitions. Stipulative definitions are used in formal logic for the (a) the initial naming of a concept that is newly-introduced (e.g. any of the elementary logical concepts newly-introduced to a student), or (b) in the notational abbreviation of one linguistic expression for another (meaningful) linguistic expression (e.g. '& = and,' 'mathematical term = precise mathematical formula'), or (c) in a precise formalization where a precise definiens alteration (or specification) is proposed for technical reasons (e.g. a number n is 'even' if and only if there exists an integer k such that $n = 2k$). All mathematical concepts are derivative from stipulative definitions.

An axiom system need not be a series of statements about a subject matter; but rather can be a system of conditions for what might be called a relational structure. Mathematicians are interested in formal derivations by which a system of mathematical concepts can be reduced to (i.e., identified with) or related to other established structures. Most often mathematical structures have value in exhibiting a definite network of properties and relations that are useful for modeling or engineering some aspect of the world.[41]

The primary purpose of a philosophy of mathematics is to interpret mathematics and illuminate the place of mathematics in the overall intellectual enterprise. A philosophy of mathematics should be judged by holistic standards, on how well it accounts for the ontology, epistemology, and semantics of mathematics. By using the method of conceptual analysis, I believe that this chapter is largely successful in achieving these basic goals.

[41] A major historical concern for philosophers is how can mathematicians have epistemic access to these structures? Is mathematics about an *a priori* real and objective order of abstract mind-independent entities or are mathematical objects entirely mind-dependent? Although the mind-dependent hypothesis is strongly advocated here, this chapter has not fully discussed these metaphysical-epistemic-ontological issues. A response to these questions will require additional chapters (thirteen and fourteen).