A Tripartite Theory of 'Definition'

Abstract: This essay analyzes the nature of 'definition' as a definiendum-to-definiens relationship. A 'tripartite theory' of definition is hypothesized. It states that whenever a person defines a definiendum-to-a-definiens, that person can only be interpreted as asserting either a 'reportive definition,' a 'theoretic definition,' or a 'stipulative definition.' In order to verify the truth of the theory, a conceptual investigation about the functional use of definitions in various situations is described by examples. Of special interest are the examples of 'stipulative definition.' As a mathematical anti-realist, I contend that formal systems are largely composed of stipulative definitions that are either 'technically formalized' or 'abbreviatory' in nature. To back up the tripartite theory, I discuss Carnap's concept of 'explication,' and sketch a 'game formalism' account of mathematics. A theory of definition and an epistemology of mathematics is presented.

Introduction

With a tripartite theory of definition, I hypothesize that whenever a person asserts how a linguistic entity (i.e., word, phrase, symbol, definiendum) has been used, is used, or is going to be used; that person can only be interpreted as asserting a reportive (i.e., lexical) definition, theoretic definition, or a stipulative definition. If this hypothesis is true, we should be able to understand any definition of a definiendum-to-definiens form (in a context) as being one of these three types. If this hypothesis is false, we should be able to find an instance of a linguistic token-to-meaning form that cannot be interpreted as reportive, theoretic, or stipulative. The tripartite theory is not an *a priori* truth; it is a social scientific conceptual truth that could be disconfirmed with counter examples.

The methodology of this essay is that of conceptual analysis. With this method, we will evaluate a theory of 'definition' as a 'best-explanation inference' about the nature and functional use of the term 'definition.' Possessing a concept (such as of 'definition') makes one disposed to have beliefs (or intuitions) about the correct application of the concept in various cases. With conceptual analysis, participants are asked to critically assess their conceptual intuitions (which are subject to clarification). We are concerned with hypotheses and functional explanations about how natural and artificial languages are used in the context of the (implicit and explicit) intentions of users.

What is a 'definition' (as a definiendum-to-definiens relationship)? To guide our pre-theoretic intuitions, let's start out with a question. Each of the seven assertions below is an example of a definition. True or false?

(1) 'Knowledge' means 'cognition, or the fact of knowing something through acquaintance, or range of one's information or understanding, or the sum total of truth, information, and principles acquired by humankind.' (A dictionary).

(2) 'Water' means 'a clear liquid that falls as rain, and makes up streams, lakes, and seas, and is composed of H2O.' (A man on the street).

(3) 'Knowledge' means that '**S** possesses a justified true belief that **p**.' (A philosopher in the 1950's).

(4) 'Water' means 'a substance composed of H2O, which freezes at zero degrees centigrade, and has a high maximum density at 4 degrees centigrade, and a high specific heat.' (A physical scientist).

(5) I shall at this moment name my new puppy 'Spot.' (A dog owner).

(6) In the remainder of this essay, I abbreviate 'trigeminal neuralgia' as 'TN.' (Author proposes a short symbol for a long symbol to save space).

(7) A person is 'tall' if he or she is 6 feet in height or greater. (A person evaluating how many tall people are playing in a high school basketball league).

I hope the reader has responded 'true' to the above question. Each sentence is an example of a definition. In each case, a linguistic expression, the *definiendum*, is the subject of the sentence. It is related as being 'equivalent' to a *definiens* which is stated with sentences or phrases that already have a meaning (i.e., an intelligibility) to the reader.

Part I: A Tripartite Theory of Definition

These examples serve as a guide to the theory of definition hypothesized here. A **'definition'** is a sentence that connects a mark or a sound (i.e., a definiendum) to a meaningful definiens in the context of the following three functions:

(1) A **'reportive definition'** (or 'lexical definition,' 'nominal definition') reports or describes the generally accepted or community equivalence between a definiendum and a definiens. A reportive definition is correct (i.e., true) if its definiens is an accurate report of the usual sense(s) of a definiendum. A standard dictionary contains reportive definitions.

(2) A **'theoretic definition'** (or 'real definition,' 'natural definition') affirms the standard equivalence between a definiendum and a definiens, but represents an attempt to analyze the 'nature' or 'associated material conditions' of the entity being discussed. Entities designated by a theoretic definition are assumed to have a self-unity, or an independent nature that allows them to have essential properties

to be the subject of analysis. In physical science, objects such as water, acid, gold, kinetic energy, electron, gene, protein, enzyme, animal species, and plant species are often thought to belong to 'natural kind' categories. In Philosophy, the concepts of knowledge, truth, justification, mentality, cause, law, necessity, identity, explanation, freedom, beauty, goodness, piety, justice, and existence have often been treated as having an objective nature, and capable of theoretic definition. A theoretic definition is correct (i.e., true) if its definiens truly describes instances of the object being defined. Attention to evidence, reasons, and arguments is required to establish the truth of a theoretic definition.

(3) A **'stipulative definition'** introduces a specialized definiens for a definiendum. This occurs in the following three contexts: (a) the initial naming of an entity where the entity is newly-discovered, newly-introduced, newly-created, or newly-renamed, or (b) in the notational abbreviation of one linguistic expression for another (meaningful) linguistic expression, or (c) in a precise formalization where a reportive definiendum-to-definiens relation is generally affirmed but a definiens alteration (or explication) is proposed for pragmatic, technical, or personal reasons.

The evidential support for the tripartite theory of definition is based upon the observations of speech and writing patterns found in natural and artificial languages. The theory should account for definitions that are found in the physical sciences, mathematics, and elsewhere. All other kinds of definitions (e.g., analytic, ostensive, real, nominal, synonymous, recursive, explicit, implicit, precising, persuasive, operational, essential, disjunctive, verbal, conventional, intensional, extensional, contextual, explicative, functional, conditional, impredicative, partial, axiomatic, constructive, procedural, direct, legislative, discursive, etc.) should be identical to, fall under, be explainable, or refutable under these three primary types. This theory is a hypothesis about the actual limits (and modes) of how persons can intelligibly specify their use of a linguistic symbol. The tripartite theory is very similar to those found in the elementary logic books of Irving Copi & Carl Cohen (2005) and Patrick Hurley (2009). The 'tripartite theory' shouldn't be controversial. My goal is not to introduce an entirely new theory, but to call attention to it. If the tripartite theory is true, then it has importance for explaining and defending an 'anti-realist' philosophy of mathematics, as well as helping resolve some other issues in analytic philosophy, including issues about language.

Although there is an ancient distinction between so-called 'real' and 'nominal' definitions, and the concept is intermittently discussed amid various philosophical inquiries, there is an absence of long-term analyses of 'definition' as a unified concept. The only book-length treatment of this topic that I am aware of is Richard Robinson's *Definition* (1954). The fact that there are few explicit theories of 'definition' is confirmed by several sources. In the December 1993 volume of *Philosophical Studies*, guest editor Marian David chose 'Definitions' as a topic for submitted articles because despite their important role in analytic philosophy "there is hardly any literature" about definition. In that same volume, Nuel Belnap (1993) is disappointed about not finding substantial modern theories of definition, especially in texts that are histories of logic.

(1) Reportive Definitions

The concept of a reportive definition is familiar. This kind needs little elaboration and only a few examples. The truth (or falsity) of a reportive definition depends upon whether the sense(s) attributed to the linguistic entity are in fact the senses attributed to the symbol by a community. The entries in any standard dictionary are examples of reportive definitions. Reportive definitions are intended to be 'true to' actual usage. For example, the dictionary definition that 'mountain' means 'a large mass of earth or rock rising to considerable height above the surrounding landscape' is a true report of how English-speaking people use the word. But dictionaries are not the only source of reportive definitions. When asked for the definition of a term, a person can report what is believed to be the ordinary meaning of the term.

(2) Theoretic Definitions

The second kind of definition is a 'theoretic definition.' A theoretic definition generally affirms the reported equivalence between a definiendum and a definiens, but further seeks to analyze the 'nature' or 'associated material conditions' with respect to a natural kind of entity. A natural kind entity is thought to have intrinsic properties and an independent nature. Aristotle, in *Metaphysics* VII and the *Posterior Analytics*, invokes a concept of ‘definition’ with respect to the natural world. Aristotle was concerned with definitions about 'substances' that he conceived to be 'naturally unified' entities, which included animals and plants. 'Substances' have self-unity or a self-contained form. At *Posterior Analytics* 93b29, he maintains that a definition is an account of 'what a thing is.' At *Metaphysics* 1031a12, he states that a definition is the formula of the essence, and the essence must belong to substances. At *Topics* 101b38, he states that a definition is a phrase signifying a thing's essence (Ross, translator). Aristotle compares other objects that do not have an intrinsic self-unity (e.g., a pile of sand, a rock, a table, a bronze statue) and calls them ‘deficient’ or ‘derivative’ in 'being' compared to substances. Natural kind entities, but not derivative beings, can be said to be the object of a theoretical definition.

Hilary Kornblith provides a scientific characterization of a 'natural kind' as a product of 'homeostatic property clusters.' A 'homeostatic relationship' is where a relatively stable state of equilibrium between interrelated physiological factors maintains even in the face of changes in environment. The concepts of homeostatic causal relationships and property clusters are also developed by Richard N. Boyd (1988, 1991). Below is text of Kornblith's (1993) account of physical natural kinds as endorsed here:

> Natural kinds involve causally stable combinations of properties residing together in an intimate relationship (p. 7) …It is nature which divides the world into kinds by creating stable clusters of natural properties residing in homeostatic relationships. Some properties are *essential* to natural kinds because they are part of this homeostatic cluster or an inevitable part of it; other properties of members of the kind are merely accidental (p. 56, italics added).

Examples of entities that purportedly possess essential properties and theoretic definitions are found in physics (e.g., electron, centripetal force, kinetic energy, heat, and torque), chemistry (acid, salt, and periodic chart elements), astronomy (black hole, planet), and psychology (intelligence, frustration). More controversially, biological terms are believed by some theorists to be natural kind concepts (gene, mice, marsupial mice, and octopus). With natural kind concepts, our attention is paid to *the (objective) nature of the phenomena* involved. Philosophical concepts such as knowledge, truth, and definition can also be conceived of as natural kinds (i.e., having unity, discreteness, and essentiality). Here are three natural kind concepts that have theoretic definitions:

(1) **'Electron'--** An electron is associated with 'quantum numbers,' 'wave-particle duality,' 'indeterminate position-momentum,' among others. Co-authors Buchwald and Warwick (2001) state that the electron's primary characteristics 'charge' and 'mass,' have become better known since its discovery in the nineteenth century. (pp. 16-17).

(2) '**Centripetal force**'—A force that acts inward on a body that rotates or moves along a curved path and directed towards the center of the path or the axis of rotation. It is measured Mass x Velocity squared, divided by Radius.

(3) **'Truth'--** The 'correspondence theory' is popular: A proposition **p** is 'true' just in case it corresponds to facts or the world. A **p** (a belief, proposition, assertion) is true if it corresponds to (or correctly describes) a state of affairs. Another definition not using the term 'correspondence' is from A.N. Prior (1971, pp. 21-22): "To say that **S**'s belief that **p** is 'true' is to say that one believes that **p** and (it is the case that) **p**."

(3) Stipulative Definitions

A 'stipulative definition' introduces a specialized definiens for a definiendum. There are three subcategories: a) initial naming definitions, b) linguistic abbreviations, and c) formalized definiens for pragmatic, technical, or personal reasons. Here are some paradigm examples of stipulative definitions (a-c, including examples c1-c3):

(a) Initial Naming Definitions

Initial naming definitions function to *introduce a new term* to denote an entity.

(1) I shall at this moment name my new puppy '**Spot**.' (Source: A dog owner declaring the name of her new puppy).

(2) This particular platinum-iridium bar (at a temperature of 0 degrees celsius) will now constitute the standard measure of a '**metre**.' (Source: The French National Academy in the late 19th century).

(3) An '**electron**' is the name designated for a negatively charged subatomic particle. (Source: Scientist, George Johnson Stoney, 1891)

(b) Abbreviatory Definitions

The function of a definitional abbreviation of the 3b variety is the *substitution* of a shorter term (the definiendum) for a longer expression (the definiens). A necessary condition of a successful abbreviatory definition is that it connects a mark or a sound (i.e., definiendum) to a *meaningful definiens*. For any definiendum-to-definiens relationship to have a cognitive intelligibility for persons involved, the definiens must have content that is (to some degree) understandable to the parties involved.

(4) In the remainder of this essay, I will abbreviate 'trigeminal neuralgia' as '**TN**.' (Source: An article about nerve disorders. The author proposes a short symbol for a longer one to save space and for easier reading).

(5) In this contract, the name 'John Smith' designates the term '**lessee**.' (Source: An apartment contract where for typographical convenience, and consistency, the predicate 'lessee' is substituted for a proper name).

(c) *Precisely Formalized Definitions*

Precisely formalized definitions involve terms that might already have an established use (and reportive definition) but where a definiens alteration is proposed for *pragmatic, technical, or personal* reasons. The function of a precise formalization is to *modify the definiens* of the *definiendum* for practical application. Considerations about 'measurement' are often involved.

(1) Pragmatically formalized definitions

(6) A person is '**tall**' if he or she is 6 feet in height or greater. (Source: A person evaluating how many tall people participate in a basketball league).

(7) '**Light**' means 'One third fewer calories.' (Source: A definition proposed by the United States Food and Drug Administration with the intent of making the labeling of food more consistent in 1991).

(2) Technically formalized definitions

(8) '**Economic equilibrium**' is a state of affairs where there is no excess demand: a state of affairs in which at the going prices nobody wants to go on exchanging. (Source: An economist).

(9) '**Abnormal behavior**' is behavior that is deviant, maladaptive, or personally distressful over a relatively long period of time. (Source: A psychologist).

(10) **'Equator'** is an imaginary line at 0 degrees latitude, 40,075 km in circumference, halfway between North and South Poles. (Source: A geographer).

(11) An **'analytic sentence'** is a sentence that is true solely in virtue of the meaning (or the definitions) of its terms. (Source: Originating with Kant).

(12) '**Truth**' is a *property* of sentences (in a given formal model) and sentences are truth bearers. (Source: Logician, Alfred Tarski, 1944).

(13) A **'proposition'** is an abstract object to which a person is related by a belief, desire, or other psychological attitude, typically expressed in a language containing a psychological verb ('think,' 'deny,' 'doubt,' etc.) followed by a that-clause. The psychological states in question are called propositional attitudes. (Source: *The Cambridge Dictionary of Philosophy*).

(14) An **'intension'** is the meaning or connotation of an expression, as opposed to its extension or denotation, which consists of those things specified by the expression. The intension of a declarative sentence is often taken to be a proposition and the intension of a predicate expression (common noun, adjective) is often taken to be a concept. (Source: *The Cambridge Dictionary of Philosophy*).

(15) An **'axiom'** is an independent foundational prescriptive assertion that underlies a set of stipulative definitions; including the vocabulary, grammar-syntax, and inference rules that measure a specified domain. Axioms cannot be deduced from other sentences in a formal system. An axiom is typically adopted if it helps map (or represent) the physical world (or linguistic discourse) in a fruitful way. (Source: This precise technical definition of 'axiom' is advocated here).

(3) Personal formalized definitions

(16) '**Happiness**' is good health and bad memory. (Source: Ingrid Bergman).

(17) '**Leadership**' is a person's being able to guide or inspire others, for support in the accomplishment of a common task. (Source: Motivational speaker).

A Comparison with Peter Geach's Similar View About Definitions

P.T. Geach (1976) similarly recognizes a difference between *real* (i.e., theoretic), *nominal* (i.e., reportive), and *proposed* (i.e., stipulative) definitions. In the following text, Geach summarizes his intuitions about the concept of definition:

> It has long been traditional to distinguish between *real* and *nominal* definitions. Real definitions aim at marking out a class of things that shall correspond to a natural kind, like gold or acids… We need, then, to recognize the natural kinds of things, and to conceptualize this recognition in a form of words describing a given kind: such is the real definition, which naturally scientists keep on updating. Nominal definition on the contrary is concerned with the use of a term. One sort of nominal definition accepts established usage, and is concerned to sort out and characterize as accurately as possible the actual uses of a word; this is the sort of definition you find in a good dictionary—though dictionaries will also contain a certain number of what would count as real definitions, of the sort just described. Another sort of nominal definition does not merely accept whatever happens to be the current usage, but constitutes a proposal for tightening up the use of a term; under the proposal, the term would mostly be applied as it now is, but with stricter criteria; or again, the proponent of the definition may suggest that we abandon some current uses and retain only one preferred use. (pp. 41-42).

With Geach's theory of definition being similar to those of Copi, Cohen, and Hurley, the tripartite theory of definition isn't really new, but it is unrecognized, and it's important to call attention to it, because it seems relevant when examining many philosophical issues.

Summary of the Tripartite Theory of Definition

In sum, I have hypothesized the following disjunctive definition for the concept of 'definition' as a natural kind entity. The definition is a *theoretic definition:*

> **x** is a '**definition**' in a definiendum-to-definiens relationship if and only if it is (1) reportive, or (2) theoretic, or (3) stipulative; (3a) an initial naming assertion, or (3b) an abbreviation, or (3c) a precise formalization for practical, technical, or personal reasons.

This definition is either *true* or *false* as a description of the nature of 'definition.' The challenge to anyone skeptical about this definition is to provide a single counter example.

Part II. On the Importance of a Theory of Definition with Respect to Mathematics

For many philosophers, a theory of definition is of little importance. In Ken Abika's *The Philosophy Major's Introduction to Philosophy* (2021), which intends to offer to a rigorous, but concise account of basic philosophical concepts for students seeking to pursue graduate study, it includes distinctions such as particulars/universals, abstract/concrete objects, singular terms/predicates, object language/metalanguage, extension/intension, properties/relations/propositions, essential/accidental properties, possible worlds, rigid designators, analytic/synthetic truths, *a priori* and *a posteriori* knowledge and truths, necessary/possible/contingent truths, propositional attitudes, and so on, but the concept of 'definition' doesn't even appear in the Index at the end of the book about topics covered. This is an oversight for the discipline of philosophy because many philosophical concepts are (in fact) stipulated as precise technical formalizations. This is important, since such stipulations are not truth-apt, but require 'acceptance.'

Carnap's Conception of 'Explication'

That many central philosophical and semantic concepts are precise technical formalizations that are 'accepted pragmatically,' was recognized by Rudolf Carnap (1950, 1956). Carnap was an adherent of ontological anti-realism. He argued that whether it is theoretically useful to employ a given linguistic framework is largely settled on pragmatic grounds. According to Carnap, the framework best suited for modeling and clarifying substantive issues (in mathematics, physics) is a matter of linguistic explication. An 'explication' was understood as the process of replacing an inexact or vague concept by an exact and precise one, ideally within the context of an artificial, precise language. It is the motivated stipulation of meanings, the setting up of frameworks with a clear semantics and well-defined rules where the internal mathematical and empirical questions could be asked and answered.

According to Carnap there was no single correct language of measurement; multiple possible languages are possible. For Carnap, the acceptance of a 'formalized redefinition' of a concept cannot be judged true or false, but it is part of the acceptance of a language where using the term will be expedient or conducive, to the measurement of a domain. A dialogue is needed among practitioners to decide what formal system and what concepts work the best to measure a domain. Carnap believed that the use of logic and definitional explication could specify the subject matter and allow for a new scientific philosophy involving piecemeal collaborative work. Philosophy could be fruitfully pursued by studying the logical syntax of the natural and artificial languages.

Carnap's permissive attitude towards the development of explications (e.g., his terms, 'intension,' 'state description,' 'L-truth') has enhanced the use of *model-building* with 'possible worlds' as a way to understand both semantics and epistemology (e.g., Timothy Williamson 2020). Nick Riemer (2010) describes logical approaches to semantics, where it is assumed that to know the meaning of a sentence, is to know what the world would have to be like, if the sentence were to be true:

> Logical approaches to semantics deal with the question of truth and reference by providing a model for the sets of logical formulae used to represent meaning. The model of a set of logical formulae is a description of a possible world to which that formulae refer, a set of statements showing what each individual constant and predicate refers to in some possible world. The model relates the logical language to this world, by *assigning referents* to each logical expression. The aim of this is ultimately to produce, for a given set of referents, a statement of the truth values of the logical formulae in which they are included. In other words, the logical formalism will tell us, given a particular world, which sentences describing this world are false and which are true. (p. 196, italics added).

A Problem for Mathematical and Metaphysical Realists

But what is the nature of '*assigning referents*' within these models? A major problem with the model-theoretic approach is that the epistemic role of the introduction of stipulative definitions as 'assigning referents' or as 'introducing specifications' of a definiens to a definiendum is ignored. It might be responded that a 'meta-language' is just describing an 'object language,' but this reply just obscures the *prescriptive* nature of stipulative definitions that are found in any language, including artificial meta-languages. Stipulative definitions are *not* truth-apt; they depend on *acceptance* by particular persons.

With the recognition of precise technical formalizations (or 'explications') and their prescriptive nature, a theory of definition is of great importance in understanding the disciplines of mathematics and formal semantics. Stipulations are neither true nor false; but can only be agreed-to. This includes the principle of excluded middle, the principle of semantic reference, and the principle of compositionality.[1] None of these principles are *a priori* true, nor can they be literally true, and all three of these stipulations may be misleading (and false)! As a mathematical anti-realist, I contend that formal systems are composed of stipulative definitions and implicit definitions (i.e., axioms).

Let us observe the structure of formal deductive systems. In formal deductive systems we typically find (1) the *stipulated* introduction of a *vocabulary* of symbols and definitions about what counts as an individual constant, individual variable, predicate, proper name, sentential connective, punctuation, and quantifier, (2) the *stipulated* introduction of *syntactical formation rules* (or grammar) that defines how 'well-formed

[1] (1) The Principle of Excluded Middle: A sentence/statement/proposition is either true or false, as a declarative sentence; in contrast to questions, exclamations, commands.

(2) The Principle of Semantic Reference: Words that are found in complete sentences and used in a context (a) refer to entities, (b) have meaning, and (c) are about something.

(3) The Principle of Compositionality: Words are the basic components of sentences, and the meaning of sentences depends (systematically) upon the meanings of the words that they are composed of.

formulas' are to be constructed out of symbols (i.e., a procedure that determines whether a sentence, as a finite strings of words or symbols, is 'meaningful' or not), (3) a set of *stipulated* truth-preserving *inference rules*, and (4) a *semantics* (e.g., truth-table definitions of connectives, or interpretations using *symbolization keys* and extensions).

The ontology of existents in the vocabulary follows from the use of stipulative definitions. Similarly, the semantics of the formal system are specified by stipulative definitions. The consistent syntactical rules and inference rules are also stipulated.

In the overview, mathematical definitions and rules can be understood as *creative constructions* used for (possibly) modeling something. The idea suggested here, following David Hilbert (1934), is that mathematicians can construct formal languages consisting of a vocabulary, syntax, consistent inference rules, and semantics without concern whether the languages are 'true' or 'correct.' Lara Alcock (2014) states an 'axiom' is a statement that mathematicians *agree* to *treat* as true (p. 9).[2] It is contended here that the adoption of axioms (as *assumed-true, not literally true*) is based upon their role in a consistent formal theory and depends upon how a theoretician constructs the theory.[3]

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[2] Moritz Schlick (1925) characterized an 'axiomatic system' as a system of truths created with the aid of implicit definitions that do not at any point rest on the ground of reality. "On the contrary, it floats freely, so to speak, and like the solar system bears within itself the guarantee of its own stability." (p. 37).

[3] Hilbert's idea of allowing symbols to remain undefined in axioms was a major break from the thought of Gottlob Frege, who believed that axioms should express objective truths, and that defined terms should have meanings that fix their denotations.

This primitive formalist perspective is in opposition to mathematical realism which contends that: (1) there exist mathematical objects, (2) mathematical objects are abstract, and (3) mathematical objects are independent of persons, including their thought, language, and practices. We will leave mathematical realism aside.[4]

From any perspective, however, it is important to observe that mathematical definitions (in fact) can be characterized as *fixed definiens concepts* (i.e., 'closed concept,' 'formal concept') with two characteristics that make up their uniqueness. First, a fixed definiens concept is a term that is stipulatively defined to *unequivocally identify* any item(s) that fall under its definition. The definiens is precise enough to distinctly exclude any entity that doesn't fall under the definition. Second, a fixed definiens concept is stable and not subject to alteration (without creating a new concept). The definiens determines what a term's proper referents (or extensions) are, if any. These terms will all have a *necessary and sufficient conditions* definition for their proper use because either (1) they have been explicitly and deliberately formulated that way (e.g., 'bachelor') or (2) were conceived to have some discoverable fixed definiens (e.g., 'limit,' 'derivative') or (3)

[4] The mathematical realist contends that mathematics is about the discoverable objective features of the world. For example, numerals denote objective numbers, and numbers are abstract and eternal. For numerals to exist, and for mathematical knowledge to exist, their propositions must be *about* something. Stewart Shapiro (2000), a mathematical realist, maintains that "we use language to talk *about* things, usually things other than language itself. In its normal usage, a symbol *symbolizes something.*" (p. 145). For Shapiro, mathematical language has 'meaning' and formalists ignore this meaning.

defined by a recursive definition (e.g., the 'successor' of ordinal number x is the next ordinal number, or x + 1), or (4) are formalized functions (e.g. (m)x is the mother of x for all persons). [5]

Part III: Mathematical Knowledge: A Sketch of 'Game Formalism'

We can now sketch a mathematical epistemology that can be labeled as 'game formalism.' On this account, mathematical knowledge can be conceived as similar to knowing the rules of a game and making moves that accord within the rules of the game. If one adopts certain rules, then there are certain valid conclusions or outputs that follow, given certain inputs. This simple epistemic-semantic view of entailed mathematical truths is similar to what has been called 'syntactic formalism,' 'deductivism,' or 'game formalism.' Besides stating that the axioms of a deductive system express implicit definitions (and primitive terms) independent of any derivation from other propositions, formalism holds that deduced mathematical 'truths' are the consequence of following a consistent set of manipulation rules. Reasoning proceeds based upon syntactically marked regularities of expressions without an immediate concern for semantics. The content of a mathematical system is exhausted by the rules operating within its language.

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[5] Frege (1903) maintains that in a formal theory "a definition of a concept (of a possible predicate) must be complete; it must unambiguously determine, as regards to any object, whether or not it falls under the concept (whether or not the predicate is truly assertable of it). Thus, there must not be any object as regards which the definition leaves in doubt whether it falls under the concept... the concept must have a sharp boundary." (Peter Geach and Max Black, eds. 1960, p. 159).

The adoption of certain concepts, definitions, and rules are typically guided by the pragmatics of measuring a given domain (e.g., numerical, spatial, valid arguments). Propositions entailed from a proof are derived relative to a system's foundations (axioms, definitions, inference rules, grammar, and vocabulary).[6]

In an interpreted formal system, logical-mathematical entities have no independent objective existence, but are stipulated to exist (i.e., invented) using definitions with a fixed definiens. Speakers do not *refer* to points, circles, numbers, and ratios as existing *objective* unified entities when talking about them; instead, speakers *use* these mathematical terms in a way that is *consistent* with stipulated definitions. Game formalism forcefully denies the existence of independent objective mathematical entities. Mathematical objects are *not* entities that (somehow) exist outside of space and time that are acausal and eternal. This formalist view is consistent with that of logician, Alonzo Church (1932), who states that "The entities of formal logic are abstractions, *invented* because of their use in describing the facts of experience or observation, and their properties, determined in rough outline by this intended use depend on their exact character on the arbitrary choice of the inventor." (p. 352).

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[6] A proof system is formed from a set of rules chained together to form proofs or derivations. Formal proofs are sequences of well-formed formulas (wffs). For a well-formed formula to be part of a proof, it might be an axiom or the product of applying an inference rule on previous wffs in the proof sequence. A symbol or a string of symbols comprise a wff if the formulation is consistent with the formation rules of the language.

With the tripartite analysis of definition, and mere sketch of a formalist theory, we have so far argued that mathematical definitions are ‘abbreviatory’ or ‘technically formalized’ (and sometimes both). We will now consider the details of this claim.

Mathematical Definitions Are Stipulated Abbreviations (the 3b form)

Mathematician Morris Kline (1967) provides an intuitive view of how 3b abbreviatory definitions work in mathematics:

> Like other studies, mathematics uses definitions. Whenever we have occasion to use a concept whose description requires a lengthy statement, we introduce a single word or phrase to replace the lengthy statement. For example, we may wish to talk about the figure which consists of three distinct points which do not lie on the same straight line and of the line segments joining these points. It is convenient to introduce the word triangle to represent this long description. Likewise, the word circle represents the set of all points which are at a fixed distance from a definite point. The definite point is called the center, and the fixed distance is called the radius. Definitions promote brevity. (p. 51).

According to mathematicians James Robert Brown (2008) and John Horty (2007), this is the 'standard' or the 'official' view of mathematical definition. The standard view maintains that definitions are constructions that are neither true nor false. Definitions posit an abbreviation of a linguistic definiendum to a linguistic definiens. Definitions are stipulated for clarity and convenience. Most often the equal sign (=), the bi-conditional sign (iff), or a definition sign (df) are used to state that the linguistic sign on the left side (i.e., definiendum) is the same (or is identical) to the content on the right (i.e., the definiens). Philosophical (and mathematical) definitions are intended to be 'neutral' in content. Definitions should play no substantive role among the premises of a deductive argument and should play no role in the outcomes of deductive proofs and arguments. Bertrand Russell (1872-1970) and Alfred North Whitehead (1861-1947) in *Principia Mathematica* (2nd edition, 1903, p. 11) endorse this standard view:

> A definition is a declaration that a certain newly introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known. It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with symbols, not with what they symbolize. Moreover, it is not true or false, being an expression of volition, not a proposition.

This standard view is assumed by Patrick Suppes in *Introduction to Logic* (1957). In a chapter entitled 'A Theory of Definition,' Suppes states that a respect for *definitions as abbreviations* is crucial when solving deductive proofs from an already specified set of adopted existents.

Here are some examples of the use of 3b definitions, found in mathematics:

(1) The symbol **'v'** is to be used to designate 'not'

(2) A number *n* is '**even'** if and only if there exists an integer *k* such that $n = 2k$.

(3) A function *f* from the set X to the reals is **'bounded above'** on X if and only if there exists M in the reals such that for all x in X, f(x) is less than or equal to M.

In many mathematical texts, these 3b definitions are presented as 'symbols' and 'meanings.' Here we refer to 'definiendums' and 'definiens'.

Mathematical Definitions Are Formalizations (the 3c form)

Let's illustrate how mathematical definitions can be viewed as non-objective 'formalizations,' 'explications' or 'creative constructions' by considering a formal language such as Euclidean geometry. The notions of point, line, and straight were natural language concepts before the work of early mathematicians. The following three sentences are examples of a 'precise formalization' of these natural language terms in the *Elements* of Euclidean geometry:

1. A 'point' is that which has no parts.
2. A 'line' is length without breadth.
3. A 'straight line' is a line that lies evenly with the points on itself.

Euclid asserted these precise definitions as part of an attempt to enunciate the smallest basic definitions that underlie the practice of geometry and arithmetic. Euclid's definitions included geometric concepts that were used by previous geometers:

1. A 'boundary' is that which is the extremity of anything.
2. A 'figure' is that which is contained by any boundaries or boundary.
3. A 'circle' is a plane figure contained by one line such that all the straight lines falling upon it from one *particular point* among those lying within the figure are equal.
4. The *particular point* in definition 3, is called the 'center' of the circle.

From an epistemic point of view, it appears that the definitions of these seven geometric entities are stipulative definitions in the 3c sense, as precise formal improvements to the preexisting ordinary language use of these concepts. Intuitively, these geometric concepts do not seem to represent independent natural objective theoretic entities. Instead, they are stipulated as a means for fruitful measurement and determining what extensions fit those specifications. These four definitions are not subject to being true (or false) but instead they are technical stipulations (explications or creative constructions) prescribed for acceptance and adoption.

Another important 3c form of fixed-definiens definitions found in mathematics is that of 'recursive definition.' A ‘recursive definition’ (also called 'inductive definition' and 'definition by recursion') is a definition in three clauses: (1) the expression defined is applied to certain items (the base clause); (2) a rule is given for reaching further items to which the expression applies (the recursive, or inductive clause); and (3) it is stated that the expression applies to nothing else (the closure clause). The characteristic features of a recursive definition: one or more clauses non-circularly define the most basic members of the set being defined, followed by one or more recursive clauses defining how other members of the set are built out of the more basic members. Below, the concept of being part of a 'family' is an example of a recursive definition:

(*x*) (*x* is in Smith's family = *x* is Smith,

or *a* is in Smith's family

And *x* is married to *a*

or *a* is in Smith's family

And *x* is born to *a*

or *a* is in Smith's family

And *x* is adopted by *a*).

A recursive method works when there is a finite number of types of basic members of the set and there are only a finite number of ways in which non-basic members can be built up or added.

A third kind of a stipulated fixed-definiens definition found in mathematics, is that of a 'function.' A simple example of a function is where we define "function (m)" so that (m)x is the mother of the person x for all persons (who are elements of a set). If Jessica Alba is semantically designated as the person (as input x), the function specifies Catherine Alba as her mother. This well-defined function assumes that each element x is mapped to a unique element y (every person x has exactly one mother y). The output (extensions) of a function is designated by its input and fixed definiens. Alfred Tarski (1946) shared the following, stating that *definitions* are:

> …conventions stipulating what meaning is to be attributed to an expression which has thus far not occurred in a certain discipline, and which may not be immediately comprehensible…For this purpose it is necessary to define a symbol first, that is, to explain exactly its meaning in terms of expressions which are already known and whose meanings are beyond doubt… every definition may assume the form of an equivalence; the first member of that equivalence, the DEFINIENDUM, should be a short grammatically simple sentential function containing the constant to be defined; the second member, the DEFINIENS, may be a sentential function of an arbitrary structure, containing, however, only constants whose meaning either is immediately obvious or has been explained previously… In order to emphasize the conventional character of a definition and

> to distinguish it from other statements which have the form of an equivalence, it is expedient to prefix it by the words such as "*we say that.*" (pp. 33- 35).

Tarski states that a new symbol when introduced as a mathematical definition must possess a meaningful definiens, before it can be committed to a theory. To define a symbol, we must first explain its meaning in terms of expressions already known. Tarski's statement that definitions are conventionally prefaced by "we say that" effectively covers both 3c formalizations and 3b abbreviations.

The Initial Defining of Fixed Definiens Concepts

Besides the 'formalization' and the ‘abbreviation’ of terms already in use, theoreticians of any discipline must be able to construct definitions with *new terms* to designate entirely new and exotic distinctions. We can imagine that in the history of mathematics, the concepts of 'prime' and 'vertex' were each: given a fixed definiens (3c), *initially named* (3a), and later abbreviated (3b) by ancient mathematicians during the construction of arithmetic and geometry:

> 1) A 'prime number' is a natural number greater than 1 that has no positive divisors other than 1 and itself.
>
> 2) The point at which two line-segments meet, is called a 'vertex.'

Although historically inaccurate, it can be conceived that these terms were introduced to represent 'fixed definiens concepts.' They are defined to unequivocally identify any item(s) that fall under its definition, exclude any entity that doesn’t fall under that definition, and are not subject to alteration (without creating a new concept).

Not All Fixed Definiens Concepts Are Defined by Initial Stipulation

Although fixed definiens concepts are always stipulatively defined to unequivocally identify any item(s) that fall under its definition, or to specify a fixed function, the definiens for an implied fixed definiens concept can be exceedingly difficult to consistently state. Carnap (1928) states that concept formation can be "intuitively projected and maintained, but there is no recognition what the thus formed concepts actually mean" (p. 306). For example, the concept of a "derivative" of a function was put to good use for nearly two centuries before it was given a precise definiens by the work of Augustine Cauchy (1789-1857) and Karl Weierstrass (1815-1897). In the history of mathematics, Carnap observes that there was difficulty in stating the precise fixed definiens definitions for “derivative” and “limit” which were initially conceptualized and formalized in a less precise and informal 3c form. A quote from Carnap:

> The inventors of the infinitesimal calculus (Leibniz and Newton) were able to answer questions concerning the derivative (the differential quotient) of common mathematical functions; for example, the derivative of the function x-cubed is the function 3x-squared. However, they could not say to what question this expression is an answer, that is, what is actually to be understood by the 'derivative' of a function. They could indicate various applications (for example the direction of the tangent) but they could not give a precise definition of the concept 'derivative.' To be sure, they believed that they knew what they meant by this expression, but they only had an intuitive notion, not a conceptual definition… However, their formulations for this definition used such expressions

> as "infinitesimally small magnitudes" and quotients of such, which, upon more precise analysis, turn out to be pseudo concepts (empty words). It took more than a century before an unobjectionable definition of the general concept of a limit and thus of a derivative was given. Only then all those mathematical results which long since been used in mathematics were given their actual meaning. (1928, pp. 306-307).

This example illustrates that from an informal stipulative definition of 'derivative', a more precise *explication* of a formal 3c definition evolved and was developed, where a 3b abbreviation for the term 'derivative' was adopted. The concept of 'limit' and other mathematical symbols were used in explicating a precise 3c definition of 'derivative.' The derivative of $f(x)$ with respect to x is the function $f''(x)$, is technically defined as an explicit mathematical formula (found in textbooks, with exotic symbolization).

For fixed-definiens concepts, such as 'derivative' and 'limit' a more precise definiens of a 3c form was sought by mathematicians that had a consistent relationship with other postulated fixed definiens concepts.

The Objectivity of Mathematical Propositions

It is suggested here that while the 'roots' of formal systems are stipulations, the derived mathematical propositions can be known as true-in-a-language as entailed truths within a presumably consistent formal system. These same propositions can also be understood as 'descriptive' and 'objectively true' when applied to *practical questions*. For example, consider the following problem presented to a high school math student:

> Edison High School has 840 students, and the ratio of students taking Spanish to the number not taking Spanish is 4:3. How many of the students take Spanish? Choice of answers: a) 280 b) 360 c) 480 d) 560 e) 630.

It is usually thought (especially by high school mathematics teachers) that there is a *single objectively true answer* to this problem, and that the answer is *knowable* to a high school student if the correct deductive reasoning is used. The teacher's belief is true. The belief that student **S** can know the objective truth of a mathematical answer (requiring deductive reasoning) is compatible with the fact that the correct answer to each of the above problems is 'descriptive' and is 'objectively true' under the following definitions:

> A '**description**' is an assertion that purports to express a correspondence (or a representation) of some state of affairs, where its correctness (or incorrectness) is *independent* of its acceptance (or non-acceptance) by particular persons.
>
> A description is **objectively true** if it expresses a correspondence (or a representation) to some state of affairs that is independent of its acceptance (or acknowledgment) by particular persons. A description is **objectively false** if it doesn't correspond to; or represent a state of affairs.

If **S** believes that the problem can be solved by using the formula 4x + 3x = 840, and solving for x which designates a number of students, and multiplying by 4 to get answer of 480, then **S** has used the proper methodology and has relevant reasons for believing answer choice c. The answer of 480 is objectively correct because its truth is a consequence of the rules, concepts, and material conditions involved in the example.

Conclusion

With a tripartite theory of 'definition' we have hypothesized three basic kinds of definition. I respectfully ask for potential counterexamples. If there are none, then the tripartite theory is a true account of how persons (in context) may specify their intended use of a linguistic entity in a definiendum-to-definiens relationship. Again, this definition is either true or false as a description of the nature of definition; and is the main concern.

With a philosophy of mathematics, its purpose is to interpret and illuminate the place of mathematics in the overall intellectual enterprise.[7] It has been argued here that *abbreviatory* and *technically formalized definitions* are the primary kinds found in mathematics. Mathematical formalism, as advocated here, holds that deduced mathematical 'truths' are the consequence of a consistent set of manipulation rules operating within its language. Of course, a detailed theory about mathematical ontology that includes discussion about mathematical realism, nominalism, and conceptualism is still needed. But this sketch provides a good start for further discussion with other philosophers and mathematicians who have some of these similar intuitions about the nature of mathematics.

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[7] Typically, the philosophy of mathematics is treated as a subdiscipline and is isolated from epistemology. Paolo Mancosu (2008) states "I think that philosophy of mathematics has to a great extent been hijacked by metaphysics… For the most part, current epistemology of mathematics has not addressed at all matters relating to fruitfulness, understanding, explanation, and other aspects of mathematical epistemology." (p. 200).

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